Coordinating Business Cycles*

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Abstract

We develop a quantitative theory of business cycles with coordination failures. Because of demand complementarities, firms seek to coordinate production and multiple equilibria arise. We use a global game approach to discipline equilibrium selection and show that the unique equilibrium exhibits two steady states. Coordination on high production may fail after a large transitory shock, pushing the economy in a quasi-permanent recession. Our calibrated model rationalizes various features of the Great Recession. Government spending, while generally harmful, can increase welfare when the economy is transitioning between steady states. Simple subsidies implement the efficient allocation.

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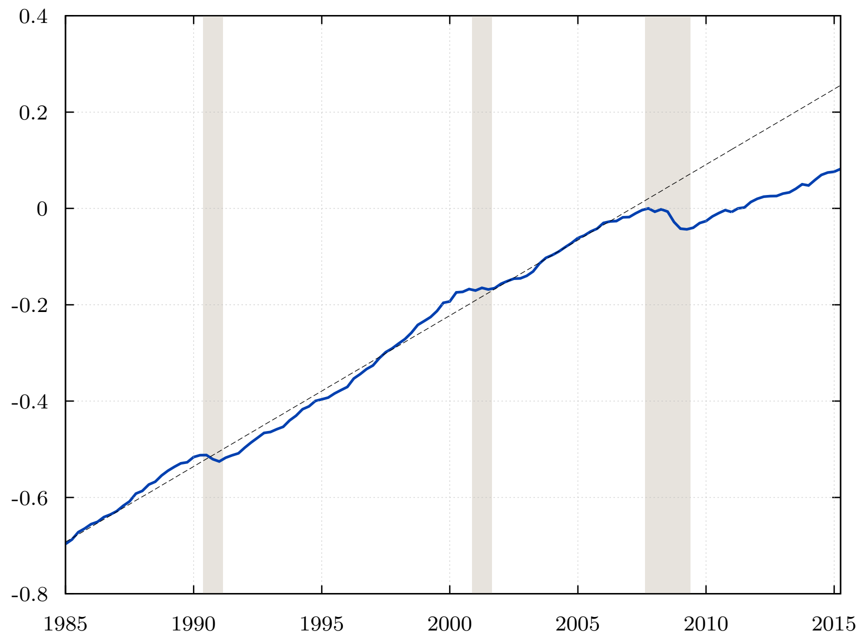
1 Introduction

During the post-war period, the United States economy has always overcome recessions by quickly reverting back to its long-run trend. In contrast, its evolution in the aftermath of the 2007-2009 recession has been startling. After the trough of the recession was reached in the second quarter of 2009, most major economic aggregates returned to growth but have not yet caught up with their previous trends. As Figure 1 shows below, real GDP seems to have settled on a parallel but lower growth path.[1]



Notes: Series shown in logs, undetrended, centered at 2007Q4. GDP is the BEA series in constant 2009 prices, seasonally adjusted. The linear trend is computed over the period 1985Q1-2007Q3. Shaded areas correspond to NBER recessions.

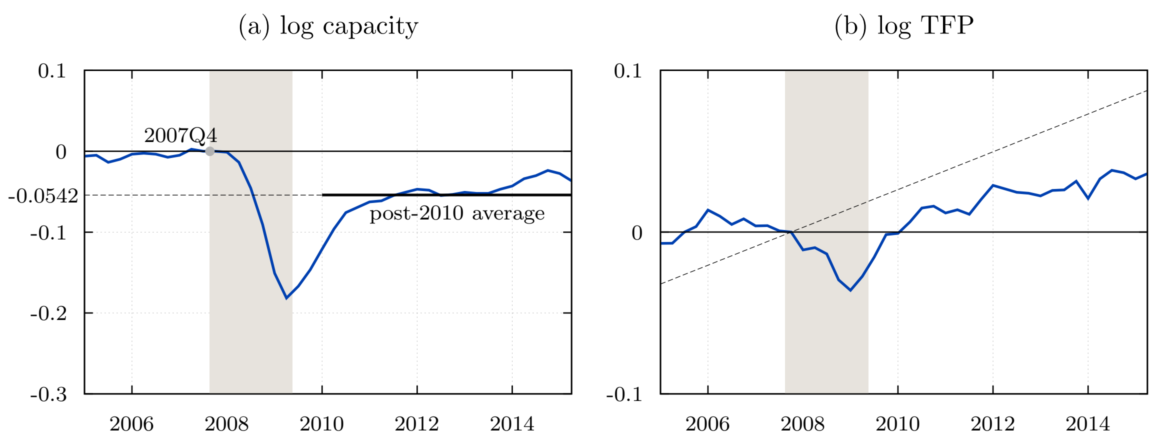
Figure 1: Evolution of US real GDP over 1985-2015

We propose a quantitative theory of coordination failures that can account for this pattern. At the heart of the mechanism are demand complementarities that link firms' production decisions: the choice by one firm to scale up production generates additional income that raises the demand for other firms' products, thereby increasing their incentives to produce. The presence of this complementarity opens up the possibility of miscoordination and multiple equilibria. We use a global game approach to discipline equilibrium selection and embed this coordination problem into an otherwise standard business cycle model. Two main insights emerge from the theory. First, as the coordination problem becomes more severe, strong self-reinforcing forces that can maintain the economy in a depressed state appear. More specifically, two steady states may arise: one with high output and high demand, the other one with low output and low demand. Sufficiently large

[1] Figure 14 in the Appendix shows that the conclusion that the economy has been performing below trend since 2007Q4 is robust to various definitions of the trend.

transitory shocks can hinder coordination on high production and trigger a transition from the high to the low steady state: the economy then becomes stuck in a quasi-permanent recession in line with the recovery from the 2007-2009 recession. Second, as our explanation for the recession relies on coordination failures, our theory suggests a role for government intervention. We study various policies and find, in particular, that government spending, while generally detrimental to coordination, may sometimes raise welfare by successfully preventing the economy from falling to the low steady state.

The theory builds on the standard neoclassical growth model with monopolistic competition. In this environment, firms are subject to a complementarity as they take into account the level of aggregate demand when making individual production and pricing decisions. This complementarity provides firms with a motive to coordinate their actions and is, as such, the first key ingredient for coordination failures to arise. The second key ingredient is the presence of a strong feedback from aggregate demand to production decisions. Following a tradition in the sunspot literature (Wen, 1998; Benhabib and Wen, 2004), we rely on variable capacity utilization to generate this feedback. This modeling choice is motivated by the data. Our theory predicts that a passage from the high to the low steady state, such as the one observed during the 2007-2009 recession, should be accompanied by a decline in capacity utilization. In line with this prediction, panel (a) of Figure 2 shows that aggregate capacity utilization fell in the course of the 2007-2009 recession and stabilized below trend. The theory also predicts that this drop in capacity utilization should be reflected in measured Total Factor Productivity (TFP). Indeed, the weak recovery has been associated with a protracted decline in TFP (Hall, 2014), as panel (b) illustrates.[2]



Notes: Series in logs, undetrended, centered at 2007Q4. Capacity utilization is the Federal Reserve Board index of Capacity Utilization. TFP is the raw TFP measure, adjusted for labor quality, provided by Fernald (2014). Linear trend for TFP computed over the period 1985Q1-2007Q3. Shaded areas correspond to NBER recessions.

Figure 2: Capacity Utilization and TFP during the Great Recession

[2]Figures 14 and 15, both in the Appendix, show that this finding is robust to various definitions of TFP and various detrending procedures.

Together, the two main ingredients of the model — the complementarity and the strong feedback created by variable capacity utilization — generate multiple rational expectation equilibria. In each period, the economy may admit a high-output and a low-output equilibrium. In the high-output equilibrium, firms operate at high capacity and aggregate employment and investment are high. On the opposite, in the low-output equilibrium, firms produce at low capacity, employment and investment are low and the economy is depressed. Bringing a model with multiple equilibria to the data requires a selection device. We choose to rely on techniques from the global game literature to obtain uniqueness. For that purpose, we introduce a small amount of incomplete information and endow firms with private signals about the state of the world. We show that a unique recursive equilibrium exists in this economy when these private signals are sufficiently informative or the fundamental sufficiently volatile.

Productivity and capital play important roles in determining how firms coordinate. An abundance of cheap capital or a high productivity fuel the growth of firms and therefore facilitate coordination on high output. Inversely, if capital is scarce and expensive or productivity is low, firms are more likely to coordinate on low output. As a result, capital accumulation interacts with the coordination problem to generate rich dynamics. By encouraging coordination on high output, an abundance of capital leads to an increase in income and investment, thereby facilitating coordination on high output in the future. Capital accumulation therefore makes coordination persistent. In particular, the dynamics of the capital stock may feature two stable steady states, a situation that can be described as a *coordination trap*. The high steady state exhibits high levels of output, aggregate demand and capacity utilization, while the low steady state features the opposite. After a bad shock of sufficient size and duration, the economy runs the risk of falling into the low steady state. It then enters a chronic state of depression as it sinks into a vicious cycle of declining capital stock and miscoordination. Only large positive shocks to productivity or policy interventions can bring the economy back to the high steady state. The theory therefore provides a foundation for long-lasting demand-deficient downturns.

We calibrate the model on the United States economy and show that it performs similarly to a real business cycle model in terms of standard deviation of major aggregates and their correlation with output. It, however, outperforms the standard model in explaining business cycles asymmetries as it generates a substantial amount of negative skewness as in the data. In addition, the simulated ergodic distributions of various aggregates are bimodal, a feature that is also roughly visible in the data. The multiplicity of steady states also generates strong non-linearities in how the economy responds to shocks. We find that for small shocks the economy reacts essentially as a standard real business cycle (RBC) model: after a brief downturn, the economy grows back to its original state. For a medium shock, however, firms may fail to coordinate on high output, leading to a decline in investment that perpetuates the downturn, but the economy eventually recovers to its initial state. Shocks are therefore amplified and propagated through the coordination mechanism even without a change in steady state. For a large shock, the coordination problem becomes sufficiently severe that the economy transitions to the low steady state, never returning to its original state.

To evaluate to what degree the theory can account for the events surrounding the Great Recession, we calibrate a sequence of productivity shocks to replicate the observed TFP series over 2007-2009 and then let productivity recover. We find that these shocks are sufficiently large to push the economy from the high to the low steady state. In addition, the time series generated by the model broadly replicate the behavior of their empirical counterparts in the aftermath of the recession with consumption, employment, observed TFP, investment, capacity utilization and output stabilizing to a lower steady state after a period of transition. Our coordination theory can therefore quantitatively explain some of the unusual features of this recession.

Coordination failures are often used to motivate government intervention, including government spending policies. In our model, the competitive equilibrium is inefficient because of monopolistic distortions and the associated aggregate demand externality, and government intervention is potentially useful. Our findings suggest that government spending, in the form of government consumption, is detrimental to welfare in most of the state space, as the coordination problem magnifies the dynamic welfare losses due to the crowding out of private investment. However, perhaps surprisingly, government spending may sometimes increase welfare. The intuition can be stated as follows. When preferences allow for a wealth effect on the labor supply, an increase in government spending puts downward pressure on wages. As a result, the cost of production declines and firms can coordinate more easily on high output. Through this channel, government spending helps coordination. To illustrate this mechanism, we proceed to a series of numerical simulations and find that government spending can increase welfare, with output multipliers as high as 3, when the economy is on the verge of transitioning into the low steady state.

Even though government spending can be welfare improving, it is always suboptimal. We thus consider the problem of a social planner in this economy and find that simple subsidies are enough to implement the efficient allocation. First, an input subsidy corrects the inefficient firm size that results from the monopoly distortions. Second, a profit subsidy makes firms internalize the aggregate demand externality on their capacity decision.

Related Literature

We rely on capacity utilization as the source of the feedback necessary to generate multiplicity. For that purpose, capacity utilization must create a sufficiently strong feedback from aggregate demand to output. While various modeling assumptions can generate this feedback, as for instance increasing returns, we opt for simplicity and assume that firms have access to a simple binary capacity choice that makes the capacity decision non-convex. Indeed, data show support for non-convexities in production processes. Micro-level data finds indeed that firms use various margins to adjust production over the business cycle, many of which involve non-convexities. In particular, a large empirical literature has documented the presence of large fixed costs in adjusting factors of production. For instance, Doms and Dunne (1998) and Cooper and Haltiwanger (2006) focus on capital adjustment while Caballero et al. (1997) consider costs to adjusting labor. Ramey (1991) estimates cost functions for six manufacturing industries and finds support for non-convexities.

Bresnahan and Ramey (1994) document important lumpiness in hours, overtime hours and plant shutdowns using weekly production data from the automobile industry. Also using data from the automobile industry, Hall (2000) provides evidence of non-convexities in capital utilization as firms adjust discretely the number of production shifts and operating plants over the cycle. We capture these various margins of adjustment in the simplest possible way through a discrete choice over capacity utilization.[3]

Our paper belongs to a long tradition in macroeconomics that views recessions as episodes of coordination failures. In a seminal paper, Diamond (1982) proposes a search model of the goods market subject to a thick market externality. The model features multiple rational expectation equilibria that can be viewed as a source of coordination failures. Kiyotaki (1988) builds a 2-period monopolistic competition model with increasing returns to generate equilibrium multiplicity. Cooper and John (1988) offer a unifying game-theoretic approach to show how complementarities in payoffs can give rise to multiple equilibria in static models. Jones and Manuelli (1992) propose a static framework to examine the key modeling features required to generate coordination failures. In contrast to these papers, we propose a full dynamic theory of coordination failures in an otherwise standard business cycle model. More closely related to our approach, Benhabib and Farmer (1994) and Farmer and Guo (1994) introduce increasing returns in a real business cycle model. Their economy admits a unique steady state but a continuum of possible equilibrium paths. Our contribution emphasizes the steady-state multiplicity that coordination problems can give rise to and their non-linear impact on the dynamics of the economy.

Importantly, a distinguishing feature of our paper is the use of a global game approach to discipline the equilibrium selection. Previous literature relied instead on sunspots. Even though they can produce rich dynamic patterns, sunspots can raise a number of methodological issues. A first problem is the the absence of a general consensus on the way to discipline equilibrium selection and evaluate these models quantitatively. Second, the impact of various policies on the economy depends crucially on how the equilibrium in selected. As sunspots usually select equilibria in an exogenous way, they are subject to the Lucas critique: ignoring the impact of policies on equilibrium selection may lead to incorrect policy recommendations. In response to these challenges, we rely on global game techniques. We choose this approach for two reasons. First, the assumption of common knowledge is arguably extreme and introducing a small amount of dispersed information is in fact consistent with the data. Second, because equilibrium selection is the outcome of agents' rational choices under incomplete information, our global game approach effectively lets the model "pick" the equilibrium. As a result, equilibrium selection is endogenous, and the model is not subject to the Lucas critique.

Our paper relates to the general global game literature from which it borrows a number of insights and techniques. The key result that departing from common knowledge may restore uniqueness in coordination games stems from the seminal articles of Carlsson and Van Damme (1993)

[3] Hansen and Prescott (2005) consider a real business cycle model with occasionally binding capacity constraint and find that it generates business cycles asymmetries.

and Morris and Shin (1998). Our paper further relates to the dynamic global game literature as in Morris and Shin (1999) and Angeletos et al. (2007). In comparison to these papers, we consider a macroeconomic application to business cycles in general equilibrium. The welfare criterion that we use to evaluate policies originates from Angeletos and Pavan (2007). Closer to our business cycle application, Chamley (1999) studies a stylized model of regime switches with complementarities in payoffs, and obtains equilibrium uniqueness through an imperfect information technique similar to a global game approach. Regime switches are infrequent because of slow learning about the fundamental. In contrast, our paper studies regime switches in an almost standard real business cycle model and obtains infrequent regime switches through the interaction of capital accumulation with coordination.[4]

Related to the dynamic global game literature are the works of Burdzy et al. (2001) and Frankel and Pauzner (2000) who resolve the equilibrium indeterminacy in dynamic coordination games by introducing time-varying payoffs and a sufficient amount of frictions to prevent agents to take action in every period. More closely related to our paper in this tradition is the work of Guimaraes and Machado (2014) who examine the impact of investment subsidies in an extension of the Frankel and Pauzner (2000) model to monopolistic competition and staggered technology choice. In their model, firms receive exogenous opportunities to change their technology according to a Calvo-type Poisson process. The persistence of regime changes in their model is governed by the slow arrival of these opportunities. In contrast, we rely on a global game approach to discipline equilibrium selection in a standard business cycle model with capital. The dynamics of regime switches in our model is driven by the interaction of capital accumulation and coordination.

Our approach is also reminiscent of the sentiment-driven business cycle literature as in the recent contributions of Angeletos and La'O (2013) and Benhabib et al. (2015). In these papers, in contrast to ours, the introduction of incomplete information leads to multiplicity of equilibria by allowing for correlation between information sets. As a result, the economy is subject to non-fundamental fluctuations. In our paper, we begin with a multiple equilibrium model and use a global game refinement to suppress all non-fundamentalness in the equilibrium. Hence, changes in fundamentals may trigger changes in coordination, but the economy is exempt from "animal spirits" or sentiment-driven fluctuations.

Finally, our paper touches upon various themes familiar to the poverty trap literature in growth theory. Murphy et al. (1989) propose a formal model of the Big Push idea that an economy can escape a no-industrialization trap if various sectors are simultaneously industrialized. In terms of the dynamics generated by the model, our paper is more closely related to Azariadis and Drazen (1990) who introduce threshold externalities in the neoclassical growth model to allow for multiplicity of locally stable steady states. Our paper relies on a demand-driven coordination problem to achieve similar transition stages in the dynamics of the economy and studies their implications for business cycles.

[4] Our work is also related to applications of global games such as Goldstein and Pauzner (2005) and others surveyed in Morris and Shin (2003).

The paper is structured as follows. Section 2 introduces the environment and presents our baseline model under complete information. Section 3 describes the incomplete information version of the model and establishes our main uniqueness result. In section 4, we calibrate the model and show that it replicates salient features of the recovery from the 2007-2009 recession. Section 5 analyzes the policy implications of the model and describes our findings on government spending. The full statements of propositions and the proofs can be found in the appendix.

2 Complete Information

We introduce the physical environment of our model, which remains the same throughout the paper. We begin under the assumption of complete information as it allows us to build intuition about the source of equilibrium multiplicity and the role of coordination in this economy.

2.1 Environment

Time is discrete and goes on forever. The economy consists of a representative household, a final good sector and an intermediate good sector. The final good is used for both consumption and investment. The intermediate goods consist of a continuum of varieties solely used for the production of the final good.

Households and preferences

The preferences of the representative household are given by

$$\mathbb{E}\sum_{t=0}^{\infty}\beta^{t}U\left(C_t,L_t\right), \tag{1}$$

where $0<\beta<1$ is the discount factor, $C_t\geqslant 0$ is consumption of the final good and $L_t\geqslant 0$ is labor. We adopt the period utility function of Greenwood et al. (1988) (GHH hereafter):[5]

$$U\left(C_t,L_t\right)=\frac{1}{1-\gamma}\left(C_t-\frac{L_t^{1+\nu}}{1+\nu}\right)^{1-\gamma},\gamma>0,\nu>0.$$

The representative household takes prices as given. It supplies capital K_t and labor L_t in perfectly competitive markets and owns the firms. It faces the sequence of budget constraints

$$P_t\left(C_t+K_{t+1}-\left(1-\delta\right)K_t\right)\leqslant W_tL_t+R_tK_t+\Pi_t, \tag{2}$$

where P_t is the price of the final good, W_t the wage rate, R_t the rental rate of capital and Π_t the profits it receives from firms. Capital depreciates at rate $0<\delta<1$.

[5]GHH preferences allow us to derive analytical expressions for many equilibrium quantities, but are not essential for our mechanism to operate. We relax this assumption in our policy exercises as the preference specification matters for the effect of fiscal policy.

Final good producers

The final good is produced by a perfectly competitive, representative firm that combines a continuum of differentiated intermediate goods, indexed by $j \in [0,1]$, using the CES production function

$$Y_t = \left(\int_0^1 Y_{jt}^{\frac{\sigma-1}{\sigma}} dj \right)^{\frac{\sigma}{\sigma-1}}, \tag{3}$$

where $\sigma > 1$ is the elasticity of substitution between varieties, Y_t is the total output of the final good and Y_{jt} denotes the input of intermediate good j. Profit maximization, taking output price P_t and input prices P_{jt} as given, yields the usual demand curve and the price of the final good,

$$Y_{jt} = \left(\frac{P_{jt}}{P_t} \right)^{-\sigma} Y_t \text{ and } P_t = \left(\int_0^1 P_{jt}^{1-\sigma} dj \right)^{\frac{1}{1-\sigma}}. \tag{4}$$

Intermediate good producers

Intermediate good j is produced by a monopolist that uses a constant returns to scale production function with capital K_{jt} and labor L_{jt},

$$Y_{jt} = Ae^{\theta_t} u_{jt} K_{jt}^{\alpha} L_{jt}^{1-\alpha}, \tag{5}$$

where $0 < \alpha < 1$ is the capital intensity and u_{jt} is capacity utilization. The productivity term Ae^{θ_t} depends on a constant scaling factor $A > 0$ and on a fundamental θ_t that follows an AR(1) process,

$$\theta_t = \rho \theta_{t-1} + \epsilon_t^{\theta}, \tag{6}$$

where $\epsilon_t^{\theta} \sim \text{iid } \mathcal{N}\left(0, \gamma_{\theta}^{-1}\right)$.

Capacity u_{jt} can either take a low value, normalized to $u_l = 1$, or a high value $u_h = \omega > 1$. Producing at high capacity incurs a fixed cost $f > 0$ in terms of the final good. We denote by $A_h(\theta_t) \equiv \omega A e^{\theta_t}$ and $A_l(\theta_t) \equiv Ae^{\theta_t}$ the effective TFP of firms with high and low capacity.

This capacity decision captures in a simple and tractable way different margins of adjustment, such as plants shutdowns and restarts or changes in the number of shifts and production lines, that firms use to adjust production over the cycle.[6] Importantly, this binary decision breaks the convexity of the cost function of the firms.[7] As a result, firms are able to expand their production swiftly in response to changes in aggregate conditions, which is crucial to sustain multiple equilibria in this economy.

Intermediate producers take the rental rate of capital R_t and the wage W_t as given. For each

[6] Because it acts as a TFP shifter, a broader interpretation of our capacity decision could include R&D, trade, etc. We focus on capacity utilization for its static nature and its relevance in our quantitative exercise.

[7] We restrict capacity to take only two values in order to apply the global game techniques. It is straightforward to include additional capacity levels or to model capacity choice as a continuous decision in the model with complete information.

capacity utilization u_i, $i \in \{h, l\}$, they solve the following static problem:

$$\Pi_{it} = \max_{Y_{it}, P_{it}, K_{it}, L_{it}} P_{it} Y_{it} - R_t K_{it} - W_t L_{it}, \tag{7}$$

subject to their demand curve (4) and production technology (5). Intermediate producer j then picks the capacity u_{jt} that maximizes its profits

$$u_{jt} = \underset{u_i \in \{u_h, u_l\}}{\operatorname{argmax}} \left\{ \Pi_{ht} - P_t f, \Pi_{lt} \right\}.$$

2.2 Equilibrium Definition

We are now ready to define an equilibrium for this economy. Denote the complete history of aggregate productivity shocks by $\theta^t = (\theta_t, \theta_{t-1}, \ldots)$.

Definition 1. *An equilibrium is a sequence of household policies* $\left\{C_t\left(\theta^t\right), K_{t+1}\left(\theta^t\right), L_t\left(\theta^t\right)\right\}_{t=0}^{\infty}$, *policies for firms* $\left\{Y_{it}\left(\theta^t\right), K_{it}\left(\theta^t\right), L_{it}\left(\theta^t\right)\right\}_{t=0}^{\infty}$, $i \in \{h, l\}$, *a measure* $m_t\left(\theta^t\right) \in [0, 1]$ *of firms operating at high capacity and prices* $\left\{P_t\left(\theta^t\right), R_t\left(\theta^t\right), W_t\left(\theta^t\right)\right\}_{t=0}^{\infty}$ *such that i) the household maximizes utility* (1) *subject to* (2)*; ii) intermediate producers solve their problem* (7)*; iii) prices clear all markets; and iv) the measure of firms* $m_t\left(\theta^t\right)$ *satisfies*

$$m_t\left(\theta^t\right) = \begin{cases} 1 & \text{if } \Pi_{ht} - P_t f > \Pi_{lt}, \\ \in (0,1) & \text{if } \Pi_{ht} - P_t f = \Pi_{lt}, \\ 0 & \text{if } \Pi_{ht} - P_t f < \Pi_{lt}. \end{cases} \tag{8}$$

Our equilibrium concept is standard. Notice that the definition introduces the equilibrium measure $m_t\left(\theta^t\right)$ of firms with high capacity, which must be consistent with individual capacity decisions (8).

2.3 Characterization

Two features of our environment simplify the characterization of the equilibria: i) under GHH preferences, the amount of labor supplied by the household is independent of its consumption-savings decision, ii) the problems of the final and intermediate good producers are static. We can therefore characterize the equilibrium in two stages: we first solve for the *static* equilibrium in every period, which determines production and capacity, and we then turn to the *dynamic* equilibrium, which uses the first stage as an input, to characterize the optimal consumption-savings decision and the dynamics of the economy.

Partial equilibrium

We first characterize the decision of intermediate producers in partial equilibrium to highlight the role of aggregate demand and factor prices in their capacity choice. Substituting the demand curve (4) in the expression for profits (7), the first-order conditions with respect to capital and labor yield

$$R_t K_{it} = \alpha \frac{\sigma - 1}{\sigma} P_{it} Y_{it} \text{ and } W_t L_{it} = (1 - \alpha) \frac{\sigma - 1}{\sigma} P_{it} Y_{it}. \tag{9}$$

Total factor expenses is therefore equal to a fraction $\frac{\sigma-1}{\sigma}$ of total sales, so that

$$\Pi_{it} = \frac{1}{\sigma} P_{it} Y_{it} = \frac{1}{\sigma} \left(\frac{P_t}{P_{it}} \right)^{\sigma - 1} P_t Y_t,$$

where we have substituted the demand curve (4). In this monopolistic setup, production decisions are linked across firms as the total income generated by the private sector affects the level of demand faced by each individual producers. As a result, profits, gross of the fixed cost, depend on the firm's *relative* price and on aggregate demand Y_t. In particular, when aggregate demand is high, firms have stronger incentives to expand. This demand linkage is the main source of strategic complementarity in our model.

We can now simplify the capacity decision to

$$u_{it} = \underset{u_i \in \{u_h, u_l\}}{\text{argmax}} \left\{ \frac{1}{\sigma} \left(\frac{P_t}{P_{ht}} \right)^{\sigma - 1} Y_t - f, \frac{1}{\sigma} \left(\frac{P_t}{P_{lt}} \right)^{\sigma - 1} Y_t \right\}, \tag{10}$$

where the individual prices are optimally set at a constant markup over marginal cost, $P_{it} = \frac{\sigma}{\sigma-1} MC_{it}$ and $MC_{it} = \frac{1}{A_i(\theta_t)} \left(\frac{R_t}{\alpha}\right)^{\alpha} \left(\frac{W_t}{1-\alpha}\right)^{1-\alpha}$ for $i \in \{h, l\}$.[8]

Expression (10) highlights the key forces that determine the choice of capacity in our environment. Firms with high capacity enjoy lower marginal costs of production and therefore sell their products at lower prices. Equation (10) tells us that, when choosing between the two capacity levels, firms compare two affine functions of aggregate demand — the one associated with high capacity having a higher slope but a lower intercept than the one associated with the low capacity. As a result, firms pick the high capacity when aggregate demand is high. Intuitively, when demand is high, firms face high variable costs in capital and labor and have strong incentives to pay the fixed amount f in order to exploit economies of scale and save on these costs. On the other hand, firms have no reason to pay the fixed cost when demand is low and total variable costs are relatively small.

General equilibrium

Under GHH preferences, we can derive analytical expressions for aggregate quantities as a function of the measure m_t of firms with high capacity.

Proposition 1. *For a given measure m_t of firms with high capacity the equilibrium output of the final good is given by*

$$Y_t = \overline{A} (\theta_t, m_t) K_t^{\alpha} L_t^{1-\alpha}, \tag{11}$$

[8] See Appendix E.1 for the full derivation.

where $\overline{A}(\theta_t, m_t) = \left(m_t A_h(\theta_t)^{\sigma-1} + (1-m_t) A_l(\theta_t)^{\sigma-1}\right)^{\frac{1}{\sigma-1}}$ *and aggregate labor is*

$$L_t = \left[(1-\alpha)\frac{\sigma-1}{\sigma}\overline{A}(\theta_t, m_t) K_t^{\alpha}\right]^{\frac{1}{\alpha+\nu}}. \tag{12}$$

The corresponding production and profit levels of intermediate firms are, for $i \in \{h, l\}$,

$$Y_{it} = \left(\frac{A_i(\theta_t)}{\overline{A}(\theta_t, m_t)}\right)^{\sigma} Y_t \text{ and } \Pi_{it} = \frac{1}{\sigma}\left(\frac{A_i(\theta_t)}{\overline{A}(\theta_t, m_t)}\right)^{\sigma-1} P_t Y_t. \tag{13}$$

Proposition 1 establishes a number of important results. We see from equation (11) that the economy aggregates into a Cobb-Douglas production function with TFP $\overline{A}(\theta_t, m_t)$. Importantly, this aggregate TFP is an endogenous object that corresponds to an average of intermediate firms' effective productivities. As a result, aggregate output increases with the measure of firms m_t, as high capacity firms operate a more productive technology.

This relationship between output and capacity is important as it completes our exposition about the nature of the complementarities in our environment: higher aggregate demand encourages firms to choose the high capacity; more firms choosing the high capacity, in turn, implies higher output and aggregate demand. Multiple equilibria arise in our environment when this two-way feedback between demand and capacity is sufficiently strong. This picture remains incomplete, however, if one ignores the role of general equilibrium effects in capacity decisions. These relative prices depend on factor prices, which are affected by the measure of high capacity firms m_t in equilibrium, since firms compete on factor markets. Whether there is strategic complementarity in capacity decisions in our setup ultimately depends on which of these two forces dominate: *complementarity* through aggregate demand linkages or *substitutability* through competition on factor markets.

Equilibrium multiplicity

Using our analytical results on equilibrium production and profits, we now characterize the static equilibrium capacity decision for some given stock of capital K_t and productivity θ_t.

Proposition 2. *Consider the following condition on parameters:*

$$\frac{1+\nu}{\alpha+\nu} > \sigma - 1. \tag{14}$$

Under condition (14), there exist thresholds $B_H < B_L$ *such that:*
i) if $Ae^{\theta_t}K_t^{\alpha} < B_H$*, the static equilibrium is unique and all firms choose low capacity,* $m_t = 0$*;*
ii) if $Ae^{\theta_t}K_t^{\alpha} > B_L$*, the static equilibrium is unique and all firms choose high capacity,* $m_t = 1$;
iii) if $B_H \leqslant Ae^{\theta_t}K_t^{\alpha} \leqslant B_L$*, there are three static equilibria: two in pure strategies,* $m_t = 1$ *and* $m_t = 0$*, and one in mixed strategies,* $m_t \in (0, 1)$*.*
If condition (14) is not satisfied, the static equilibrium is always unique.

Multiple equilibria arise under condition (14).[9] In regions of the state space where capital is abundant and productivity θ_t is high, such that $Ae^{\theta_t}K_t^\alpha \geqslant B_H$, a *high equilibrium* exists in which all firms choose the high capacity, $m_t = 1$. In these regions, renting capital is inexpensive and technology is productive, so firms operate at a large scale. As a result, total output and aggregate demand are high, which further encourages firms to expand and adopt the high capacity. On the opposite, in regions of the state space where capital is scarce and productivity low, such that $Ae^{\theta_t}K_t^\alpha \leqslant B_L$, a *low equilibrium* exists with $m_t = 0$: firms operate at a small scale and do not find it worthwhile to pay the fixed cost f to expand their production. For the intermediate region $B_H \leqslant Ae^{\theta_t}K_t^\alpha \leqslant B_L$, the two equilibria coexist in addition to a third mixed equilibrium. The economy is then subject to self-fulfilling prophecies: depending on firms' expectations, it may end up in either the high or the low equilibrium. Figure 3 depicts the situation described in the proposition.

The condition for multiplicity (14) characterizes the conflict between the strategic substitutability from competition in the factor markets, on the left-hand side, and the demand-side complementarity, captured by σ. This condition is satisfied when the intermediate good varieties are strong complements, if σ is low, or when the left-hand side is large. The latter term, $\frac{1+\nu}{\alpha+\nu}$, is the elasticity of aggregate production with respect to changes in TFP and it captures the scalability of the economy to changes in average capacity. Multiple equilibria are thus more likely to arise when the scalability is high, which happens when the labor supply is elastic (ν small) and when production is intensive in the flexible factor, labor (α small). This scalability term captures, in particular, the idea that multiple equilibria can only be sustained if factor prices react moderately to changes in m_t. We assume that condition (14) is satisfied from now on.

Efficiency

At this stage, it is natural to wonder whether a planner should intervene to improve the outcome of the coordination game. We consider the following planning problem

$$\max_{K_{t+1},L_t,m_t} \mathbb{E}\sum_{t=0}^{\infty}\beta^t U\left(\left(m_t Y_{ht}^{\frac{\sigma-1}{\sigma}} + (1-m_t)\,Y_{lt}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} + (1-\delta)\,K_t - m_t f - K_{t+1}, L_t\right),$$

subject to the production function (5) and the resource constraint. Proposition (3) describes the efficient allocation.

Proposition 3. *If $\frac{1+\nu}{\nu+\alpha} > \sigma - 1$, there exists a threshold B_{SP}, with $B_{SP} \leqslant B_L$, such that the planner makes all firms use the high capacity, $m_t = 1$, if $Ae^{\theta_t}K_t^\alpha \geqslant B_{SP}$ or the low capacity, $m_t = 0$, if $Ae^{\theta_t}K_t^\alpha \leqslant B_{SP}$. The threshold B_{SP} is lower than B_H for σ small.*

Interestingly, there is an equivalence between condition (14), the multiplicity of equilibria and

[9] The multiplicity condition can be weakened by allowing for decreasing returns. For aggregate returns to scale $0 < \eta < \frac{\sigma}{\sigma-1}$, the condition for multiplicity becomes $\frac{1}{\eta}\frac{1+\nu}{\alpha+\nu} > \sigma - 1$.

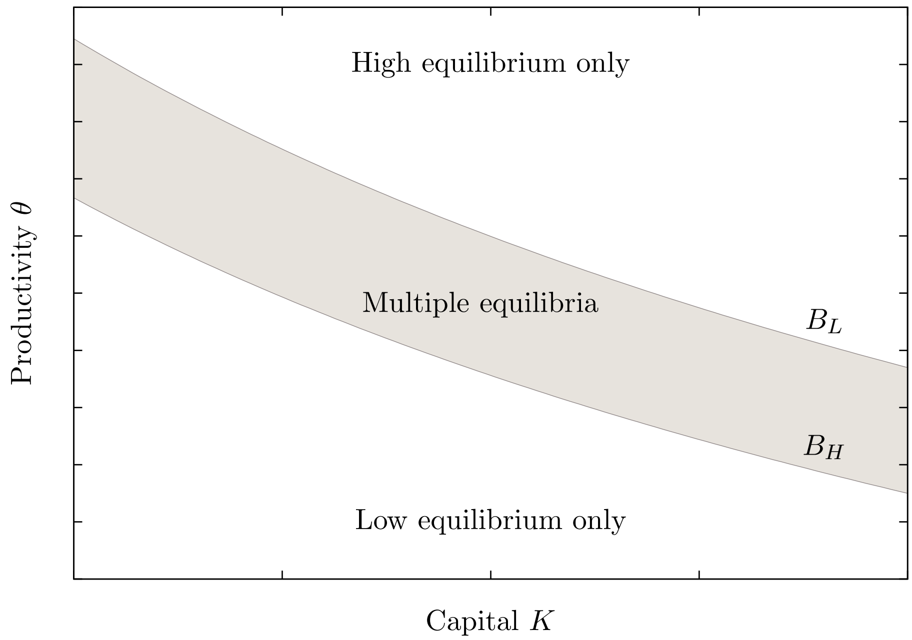


Figure 3: Multiplicity in the static game as a function of the state space

the convexity of the planner's problem in m_t. Therefore, when (14) is satisfied, the planner always chooses a corner solution, either $m_t = 0$ or 1. Since coordinating on the high capacity is costly, the planning solution is non-trivial and there exists a threshold B_{SP} such that all firms adopt the high capacity if and only if $Ae^{\theta_t}K_t^{\alpha} \geqslant B_{SP}$. When the productivity level and the capital stock are low, using the high capacity is too expensive and it is efficient to coordinate firms on the low equilibrium.

Because of the demand externality, the efficient allocation differs in important ways from the competitive outcome. Figure 4 shows the social planner's (SP) threshold, B_{SP}, together with the thresholds of the competitive economy (CE), B_L and B_H. Proposition 3 shows that B_{SP} always lies below B_L which indicates that the planner is more prone to pick the high capacity. This result is a direct consequence of the demand externality: firms do not internalize that by choosing the high capacity, they would generate more income to be spent on other firms' products, while the planner does. The competitive equilibrium therefore suffers from *coordination failures*: in the area surrounded by the dashed curves, between B_L and B_{SP}, the planner always picks the high capacity while firms in the competitive economy may coordinate differently.

Figure 4 depicts a situation in which B_{SP} lies below B_H, which happens when the degree of complementarity is strong (σ low). When σ is large, the planner's threshold B_{SP} lies between B_H and B_L and, in part of the state space, the planner may sometimes prefer the low capacity when the competitive economy could coordinate on the high equilibrium.

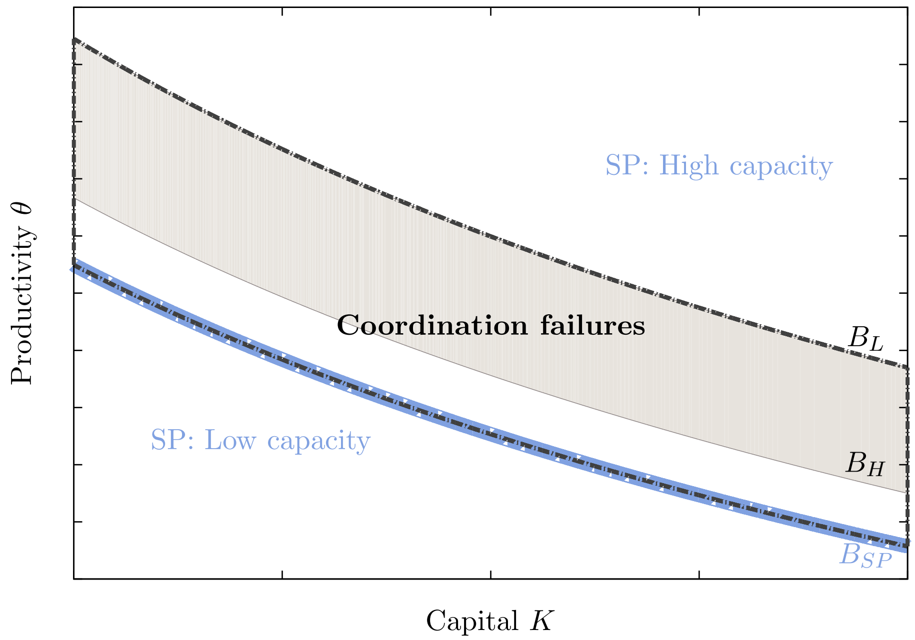


Figure 4: Planner's decision versus the outcome in the competitive economy.

3 Incomplete Information

The forces that lead to multiplicity in the model with complete information may have interesting dynamic implications, but the presence of multiple equilibria raises important methodological issues for policy and quantitative analysis. This multiplicity is, however, fragile and hinges on the assumption of common knowledge. In this section, we adopt a global game approach. By introducing incomplete information in the model, we show that uniqueness of the full dynamic general equilibrium obtains for a small departure from common knowledge.

3.1 Environment

To cast the model into a global game framework, we slightly modify the timing of events and the information available to firms when they choose their capacity utilization. The physical structure of the environment remains the same as in the previous section.

Information and timing

Each period t is now split into two stages: *i)* intermediate producers first choose their capacity under incomplete information about current productivity θ_t, *ii)* the true state of θ_t is then revealed, production decisions take place and all markets clear.

In the first stage, all agents know the past realizations of θ, which are included in their information set $\mathcal{I}_t = (\theta_{t-1}, \theta_{t-2}, \ldots)$. At the beginning of the period, nature draws the new productivity level θ_t from the stochastic process (6) but it remains unobservable to agents. The ex-ante beliefs of agents about current productivity are therefore $\theta_t \mid \mathcal{I}_t \sim \mathcal{N}\left(\rho\theta_{t-1}, \gamma_\theta^{-1}\right)$. In contrast to the model with complete information in which agents observed the fundamental θ_t perfectly, we assume that each intermediate producer j only receives a noisy signal $v_{jt} = \theta_t + \varepsilon_{jt}^v$ where the noise $\varepsilon_{jt}^v \sim \mathcal{N}\left(0, \gamma_v^{-1}\right)$ is iid across agents and time. After observing their private signal, firms use Bayes' rule to update their beliefs to

$$\theta_t|\mathcal{I}_t, v_{jt} \sim \mathcal{N}\left(\frac{\gamma_\theta\rho\theta_{t-1} + \gamma_v v_{jt}}{\gamma_\theta + \gamma_v}, \frac{1}{\gamma_\theta + \gamma_v}\right). \tag{15}$$

Intermediate producers then use these individual beliefs to make their capacity decisions in the first stage of the period.

In the second stage, consumption-savings decisions are made, production takes place and all markets clear. The observation of production and aggregate prices reveals the aggregate productivity θ_t, which becomes common knowledge. Since the input choices and production take place simultaneously, these decisions are made under complete information. As a result, the equilibrium expressions derived in proposition 1 are still valid, with the exception that m_t is now the solution to the coordination game under incomplete information that we describe below. After observing the true value of θ_t, the private signals are no longer useful and are discarded. Firms therefore share the same information at the beginning of every period.

Capacity decision

Under the new information structure, the surplus from using the high instead of the low capacity is the difference between the expected profits from using both capacities:

$$\Delta\Pi\left(K_t, \theta_{t-1}, m_t, v_{jt}\right) \equiv \mathbb{E}_\theta\left[U_c\left(C_t, L_t\right)\left(\Pi_h\left(K_t, \theta_t, m_t\right) - f - \Pi_l\left(K_t, \theta_t, m_t\right)\right) \mid \theta_{t-1}, v_{jt}\right]. \tag{16}$$

An agent with private signal v_j chooses high capacity if and only if $\Delta\Pi\left(K_t, \theta_{t-1}, m_t, v_{jt}\right) \geqslant 0$. Three important features of expression (16) are worth emphasizing. First, in contrast to the complete information case, agents compute the expectation of profits under their own individual beliefs, given by (15). Second, in addition to the uncertainty about the fundamental θ_t, there is strategic uncertainty in this environment: since other agents base their decisions on their own noisy private signals, the measure of firms using the high capacity is itself uncertain and m_t is a random variable. Third, because of the uncertainty within the period, between stage 1 and 2, intermediate producers take into account the fact that the household does not value consumption equally in all states of the world. As a result, firms use the representative household's stochastic discount factor $U_c\left(C, L\right)$ to evaluate profits.

Equilibrium definition

Because the global game selects equilibria as a function of $\left(K_t, \theta_{t-1}\right)$, the economy has a Markovian structure. We thus define a recursive equilibrium for this economy. We use θ_{-1} to denote the productivity of the previous period and normalize the price index to $P_t = 1$ in each period.

Definition 2. *A recursive equilibrium consists of i) a value function for the household $V(k; K, \theta, m)$ and decision rules $\{c(k; K, \theta, m), l(k; K, \theta, m), k'(k; K, \theta, m)\}$; ii) decision rules for individual intermediate producers $\{Y_i(K, \theta, m), K_i(K, \theta, m), L_i(K, \theta, m), \Pi_i(K, \theta, m)\}$ for $i \in \{h, l\}$; iii) aggregates $\{Y(K, \theta, m), L(K, \theta, m), \Pi(K, \theta, m)\}$; iv) price schedules $\{R(K, \theta, m), W(K, \theta, m)\}$; v) a law of motion for aggregate capital $H(K, \theta, m)$; and vi) a measure $m(K, \theta_{-1}, \theta)$ of firms with high capacity such that:*

1. *The household solves the problem*

$$V(k; K, \theta, m) = \max_{c,l,k'} U(c, l) + \beta \mathbb{E}\left[V\left(k'; H(K, \theta, m), \theta', m'\right) | \theta\right]$$

   subject to $c + k' - (1 - \delta) k \leqslant R(K, \theta, m) k + W(K, \theta, m) l + \Pi(K, \theta, m)$;

2. *Intermediate producers of type $i \in \{h, l\}$ solve the problem*

$$\Pi_i(K, \theta, m) = \max_{P_i, Y_i, K_i, L_i} P_i Y_i - R(K, \theta, m) K_i - W(K, \theta, m) L_i,$$

   subject to $Y_i = P_i^{-\sigma} Y(K, \theta, m)$ and $Y_i = A_i(\theta) K_i^{\alpha} L_i^{1-\alpha}$;

3. *Aggregate output and profits are given by*

$$Y(K, \theta, m) = \left(m Y_h(K, \theta, m)^{\frac{\sigma-1}{\sigma}} + (1 - m) Y_l(K, \theta, m)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}},$$
$$\Pi(K, \theta, m) = m(\Pi_h(K, \theta, m) - f) + (1 - m) \Pi_l(K, \theta, m);$$

4. *Capital and labor markets clear*

$$K = m K_h(K, \theta, m) + (1 - m) K_l(K, \theta, m),$$
$$l(K; K, \theta, m) = m L_h(K, \theta, m) + (1 - m) L_l(K, \theta, m);$$

5. *Consistency of individual and aggregate capital decisions: $H(K, \theta, m) = k'(K; K, \theta, m)$;*

6. *The aggregate resource constraint is satisfied*

$$c(K; K, \theta, m) + H(K, \theta, m) = Y(K, \theta, m) + (1 - \delta) K - m f;$$

7. *For all K, θ_{-1} and θ, the measure of firms with high capacity $m(K, \theta_{-1}, \theta)$ solves the fixed point problem*

$$m(K, \theta_{-1}, \theta) = \int \mathbb{1}\left[\Delta\Pi(K, \theta, m, v_j) \geqslant 0\right] \sqrt{\gamma_v} \phi\left(\sqrt{\gamma_v}(v_j - \theta)\right) dv_j, \qquad (17)$$

   where ϕ is the probability density function of a standard normal and $\Delta\Pi$ is defined by (16).

Our definition of a recursive equilibrium is standard except for condition (17) which corresponds to the equilibrium of the global game played by the firms: the measure m is the aggregation of capacity decisions when individual firms have the correct beliefs about its equilibrium distribution.

3.2 Existence and Uniqueness

When choosing their capacity utilization, firms play a global game as in Carlsson and Van Damme (1993) and Morris and Shin (1998). A key insight from this literature is that the existence of multiple equilibria depends on the information structure. In particular, full knowledge about the strategy of the other players allows agents to coordinate in a way that leads to multiplicity. The introduction of a small amount of strategic uncertainty, however, can eliminate this multiplicity. We extend these results to our dynamic general equilibrium environment.[10]

Another contribution of this paper is to show how uniqueness of the static capacity-decision game extends to the rest of the dynamic environment. This result is not a straightforward application of standard global game techniques for several reasons. First, there is a complex two-way feedback between the game and the dynamic consumption-savings choice. Second, firms' capacity decisions aggregate into a non-concave production function with endogenous TFP. Third, our economy is subject to distortions due to monopolistic competition. All these factors require specific techniques to prove the uniqueness of the equilibrium.

We now state our main result.

Proposition 4. *For γ_v large and ω sufficiently close to 1, such that, in particular,*

$$\frac{\sqrt{\gamma_v}}{\gamma_\theta} > \frac{1}{\sqrt{2\pi}} \frac{\omega^{\sigma-1} - 1}{\sigma - 1}, \tag{18}$$

and additional assumptions stated in the Appendix, there exists a unique dynamic equilibrium. The equilibrium capacity decision takes the form of a continuous cutoff $\hat{v}\left(K, \theta_{-1}\right)$ such that firm j invests if and only if $v_j \geqslant \hat{v}\left(K, \theta_{-1}\right)$. Furthermore, the cutoff is a decreasing function of its arguments.

The proof of proposition 4 is structured according to the natural separation that arises in our model between the static capacity-production stage and the dynamic consumption-savings stage. In the first part of the proof, we focus on the global game, taking some stochastic discount factor as given, and provide sufficient conditions for the uniqueness of the static equilibrium of the game. However, uniqueness of the static coordination game is not sufficient to guarantee uniqueness of a dynamic equilibrium because there is complementarity across periods. In the second part of the proof, we show that the economy under the endogenous TFP that arises from the global game admits a unique dynamic equilibrium.

[10] Applications of global games to market economies are sometimes problematic as prices may reveal enough information to restore common knowledge and multiplicity (Atkeson, 2000). In our setup, prices do reveal the true value of the fundamental, but since they are only determined at the production stage, after the capacity decisions are taken, we retain the uniqueness result. With simultaneous capacity and production decisions, uniqueness may still obtain as long as prices and aggregates are not fully revealing, for instance because of noise traders, decentralized trading, etc. See Angeletos and Werning (2006).

Part 1 proceeds in two steps. First, we show that when private signals are sufficiently precise, i.e., γ_v large, consumption risk vanishes and we can ignore the stochastic discount factor in expression (16). This step is particularly useful as it allows us to approximate arbitrarily well the solution to the global game by solving a simplified game independently from the consumption-savings decision of the household.

As is common in the global game literature, this game is solved by iterated deletion of dominated strategies as in Morris and Shin (1998). Strategic uncertainty is essential for this procedure. In particular, higher strategic uncertainty leads to more substantial deletion of strategies at each iteration, which promotes uniqueness. Condition (18) is sufficient to guarantee that the deletion process converges to a unique equilibrium, which takes the form of a cutoff strategy. It states in particular that the fundamental θ must be sufficiently uncertain (γ_θ small) and, perhaps surprisingly, that private signals must be sufficiently precise (γ_v large). This last condition is required to generate enough strategic uncertainty: since firms put more weight on their heterogenous signals when they are precise, γ_v must be sufficiently large to generate enough dispersion in beliefs and, therefore, in strategies.

Part 2 of the proof deals with the consumption-savings problem of the household. Once the capacity decisions have been made, the model reduces to a neoclassical growth model with monopoly distortions and endogenous TFP. Because of these two features, specific techniques are required to show existence and uniqueness of the equilibrium. We show that the Euler equation admits a unique positive fixed point by exploiting its monotonicity. Our proof builds on the lattice-theoretic work of Coleman and John (2000) and Datta et al. (2002), and uses a version of Tarski's fixed-point theorem on lattices, which states that monotone operators on lattices have a non-empty set of fixed points. The proof proceeds by showing that the Euler equation is a well-defined monotone operator. Monotonicity requires ω to be sufficiently small so that aggregate production, net of fixed costs, is increasing in capital and the equilibrium interest rate decreases with K.[11] The pseudo-concavity of the Euler equation then leads to uniqueness. Our proof extends earlier work to our setup and, in particular, to endogenous TFP and GHH preferences.

According to proposition 4, the optimal capacity decision takes the form of a cutoff $\hat{v}(K, \theta_{-1})$ such that only firms with private signals $v_j \geqslant \hat{v}(K, \theta_{-1})$ produce at high capacity. Hence, the measure of firms operating at high capacity is $m(K, \theta_{-1}, \theta) = 1 - \Phi\left(\sqrt{\gamma_v}\,(\hat{v} - \theta)\right)$. Since the cutoff is decreasing in K, the equilibrium measure of firms with high capacity increases with K.

Figure 5 compares the equilibrium aggregate output $Y(K, \theta, m)$ under incomplete information to the three possible equilibria of the complete information model ($m = 1$, $m = 0$ and the mixed equilibrium). As the figure illustrates, the global game tends to "select" the low equilibrium when the stock of capital is small and the high equilibrium when it is large with a gradual transition in the shaded region where multiplicity prevailed under complete information. This gradual transition is due to the progressive adoption of high capacity by firms with dispersed beliefs. While the

[11] These properties are always satisfied in a neoclassical model, but may fail in our model if the adoption of high capacity is too abrupt in K. The condition that ω is close to 1 is sufficient to ensure that the transition is smooth enough. See Appendix F for details.

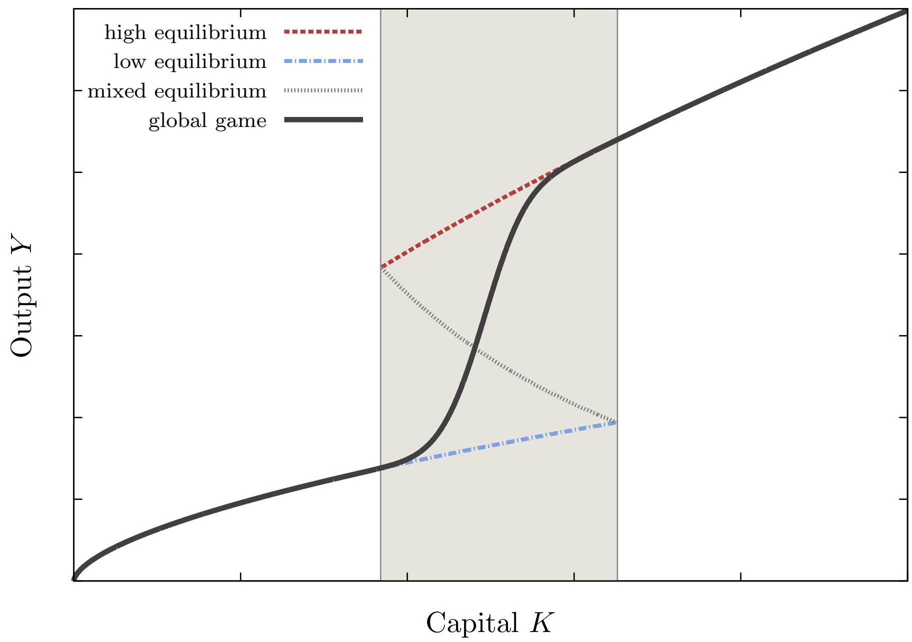


Figure 5: Aggregate output as a function of K for some given (θ_{-1}, θ)

outcomes are similar in the non-multiplicity regions, the equilibrium selected by the global game in the shaded region differs quite substantially from its complete information counterparts. In particular, the global game leads to a capacity level that increases in the economy's fundamentals, in contrast to the mixed strategy equilibrium.[12]

3.3 Dynamics

We now explore the dynamic properties of the economy under incomplete information. As was mentioned before, the model aggregates into a neoclassical growth model with an endogenous TFP that breaks the convexity of the production set. Because of this non-convexity, aggregate output Y is an S-shaped function of capital K, as shown in Figure 5. Intuitively, when capital is scarce firms prefer to operate at a low scale and, therefore, to produce at low capacity. As capital becomes more abundant, the lower rental rate increases the incentives to use the high capacity, which are further magnified by the adoption of the high capacity by other firms through the demand externality. The steep part of the S-shaped curve corresponds to the transition between the low and the high capacity.

Aggregate quantities such as consumption, employment and, importantly for the dynamics, investment inherit this S-shaped relationship to capital. Figure 6 displays the laws of motion of capital for various values of productivity θ. As the figure illustrates, for a high θ, the law of

[12] In the mixed-strategy equilibrium firms must be indifferent between using the high and the low capacity. When the capital stock is low, a large fraction of firms must therefore use the high capacity for indifference to hold. Output is consequently decreasing in capital in this equilibrium.

motion of K intersects the 45°-line once at a high capital level to the right of the transition region. Similarly, when productivity is low, the only intersection occurs at a low level of capital to the left of the transition region. However, for intermediate values of productivity, the law of motion features three intersections: a high and a low stationary point, both stable, and an unstable one in the middle region. We denote by $\overline{K}_h(\theta)$ the set of stationary points at the right of the transition region, where most firms operate at high capacity and production is high, and refer to their basin of attraction as the *high regime*. Similarly, $\overline{K}_l(\theta)$ designates the set of stationary points at the left of the transition region, where firms mostly produce at low capacity and production is low, and refer to their basin of attraction as the *low regime*. As we will see, this multiplicity of stationary points generates non-linear dynamics.[13]

The phase diagram in Figure 7 summarizes the dynamics of the economy over the whole state space.[14] The two black lines represent the high and low stationary points in the dynamics of capital: $\overline{K}_h(\theta)$ and $\overline{K}_l(\theta)$. The basin of attraction of the high stationary points in the upper right region — the high regime — is indicated by the white area, while that corresponding to the low stationary points in the lower left region — the low regime — is represented by the shaded area. Notice that the low regime does not exist for high values of θ while the high regime disappears for low values of θ.

In the absence of productivity shocks, the economy converges towards the steady state which corresponds to the basin of attraction it belongs to. Exogenous shocks to productivity θ can however push the economy from one regime to the other. When this happens, the economy starts converging towards its new steady state and the average capacity utilization adjusts accordingly.

Consider, for instance, an economy that starts at point O in Figure 7. Without shocks to θ, this economy would simply reach the high-regime stationary locus at $\overline{K}_h(0)$ and remain there. Small temporary shocks to θ can move the economy up or down on the diagram but, as long as it does not leave the high regime, the system eventually converges back to the same stationary point. A large negative shock to θ, such as the one illustrated by the dashed line from point O to O', could however move the economy to the low regime. When this happens, the low productivity level pushes firms to adopt the low capacity, leading to a low level of output. As a result, the household invests less and the capital stock declines. Coordination on high capacity is further impeded as capital falls: firms continue to operate at low capacity, perpetuating the decline in capital. As capital declines and productivity recovers, the economy follows the curved arrow in Figure 7 from point O' to the low regime stationary locus $\overline{K}_l(0)$ where it remains trapped, even after productivity has returned to normal. As we can see, the response of the economy as a function of the size of the shock is highly non-linear.

While we focus on productivity shocks for simplicity, the mechanism that we propose is by no means restricted to these shocks, but can accommodate and provide propagation to other types of

[13]In Fajgelbaum, Schaal, and Taschereau-Dumouchel 2015, we show how social learning and irreversibilities in investment can generate a dynamic system with similar properties.

[14]The full state space is (K, θ_{-1}, θ). For simplicity, we however omit θ_{-1} in our phase diagram as it becomes irrelevant in the case of interest when γ_v is large.

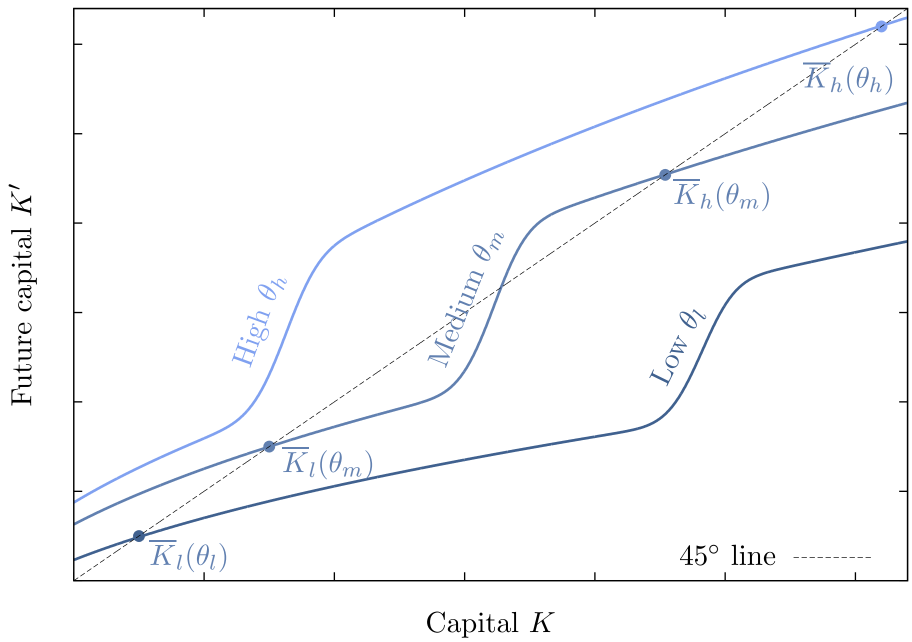


Figure 6: Multiple steady states as a function of θ

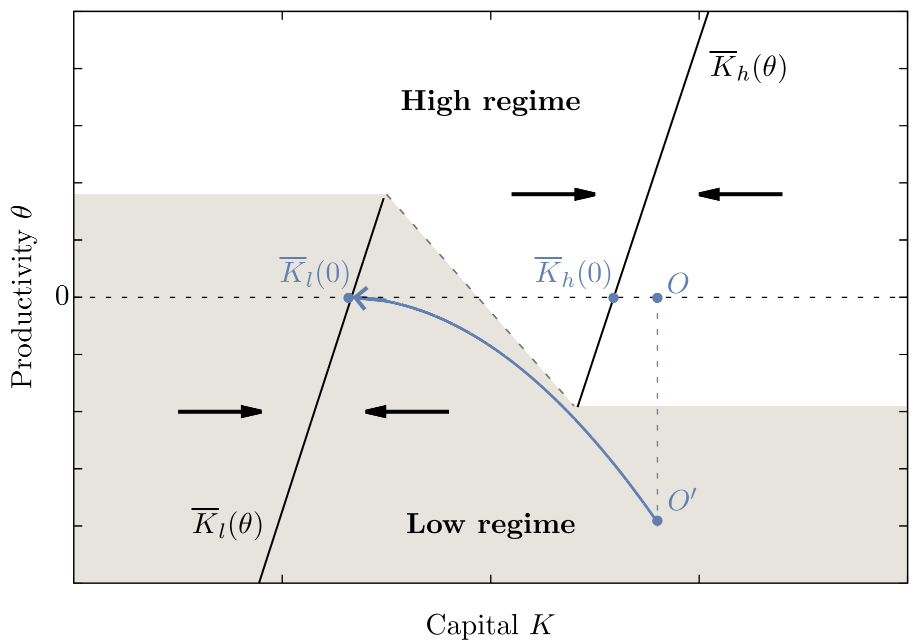


Figure 7: Phase diagram with basins of attraction

shocks considered in the literature, such as financial shocks. Consider, for instance, capital quality shocks, as in Gertler and Karadi (2011) and Gourio (2012). From Figure 7, we see that such a shock could move an economy from point O to the basin of attraction of the low regime, leading to a permanently depressed economy. In contract, in an RBC model, the high marginal product of capital that would result from this shock would lead to an increase in investment and bring the economy back to its unique steady-state. The type of coordination problem that we analyze should be considered as a general propagation mechanism, which could interact in interesting ways with other types of shocks and frictions.

4 Calibration

To evaluate the quantitative importance of coordination for business cycle fluctuations we calibrate the model to the United States economy. After analyzing the model's predictions along various business cycle moments, we run a counterfactual experiment in which we study whether the model can account for the behavior of the economy after the 2007-2009 recession.

4.1 Parametrization

Because of changes in trend growth rate before 1985, we target moments from the 1985Q1-2015Q2 period.[15] Our calibration strategy relies on the interpretation that the US economy was in the high regime over the period 1985Q1-2007Q3 and fell to the low regime after the 2007Q4-2009Q2 recession.[16] Our final quantitative exercise will justify this interpretation.

We calibrate the model at a quarterly frequency. The capital share α, the discount rate β and the depreciation rate δ are set to standard values. For the preferences of the household, we use log utility, so that $\gamma = 1$, and follow Jaimovich and Rebelo (2009) in setting $\nu = 0.4$, implying a Frisch elasticity of 2.5 in line with macro-level estimates. The fundamental productivity process θ is parametrized to replicate a persistence and a long-run standard deviation of log output of of 0.995 and 6%, as observed over 1985Q1-2015Q2. We are left to calibrate four parameters: σ, γ_v, ω and f.

The literature uses a wide range of values for the elasticity of substitution σ. In our benchmark calibration, we set $\sigma = 3$ so that the model lies in the equilibrium multiplicity region, as defined by condition (14). This parameter zone broadly corresponds to the region in which multiple steady states exist in the incomplete information environment. While it implies high mark-ups, such an elasticity is not uncommon in plant-level studies. Hsieh and Klenow (2014) uses the same value to study the life cycle of plants in India and Mexico. An elasticity of 3 also corresponds to the median estimates of Broda and Weinstein (2006) at various levels of aggregation. Using trade data, Bernard et al. (2003) estimate a value of $\sigma = 3.79$ in a model of plant-level export behavior.

[15]Figure 14 in the Appendix shows how the trend varies over different periods for GDP and TFP. Data sources are detailed in Appendix B.

[16]Following this interpretation, we detrend the log time series using a linear trend computed over 1985Q1-2007Q3.

Finally, among macroeconomic studies, Christiano et al. (2015) estimate a New-Keynesian model with financial friction and find an elasticity of 3.78. To verify the robustness of our results, we provide a full calibration of the model with $\sigma = 5$ in Appendix C. Our simulations show that the interaction of complementaries with capacity choices still generates substantial persistence in response to large enough temporary shocks, even though the multiplicity in steady states may disappear with higher values of σ.

To calibrate the precision of the private signals γ_v, which governs the dispersion of beliefs, we rely on forecasting data from the Survey of Professional Forecasters (SPF). We target the interquartile range of forecasts about current quarter log GDP which averages to 0.24% over 1985Q1-2015Q2.[17]

To calibrate the productivity gain ω from using the high capacity, we use the Federal Reserve Board index of capacity utilization series. This data is constructed using various surveys of capacity utilization rates, in particular the Quarterly Survey of Plant Capacity from the Census Bureau, and seeks to measure the ratio of current output on "sustainable maximum output". In the context of our model, we interpret this definition as capturing the change in output resulting from a change in utilization u, after allowing the firm to adjust its input choice. Taking aggregate demand and factor prices as given, Proposition 1 tells us that the ratio of individual high-capacity output on low-capacity output resulting from an individual firm deviation is $Y_h/Y_l = \omega^\sigma$. As Panel (a) of Figure 2 showed in the introduction, capacity utilization was stable before 2007, fell substantially in the midst of the recession, but rebounded quickly to stabilize at about 5% below its pre-recession level. Focusing on the recovery period, the difference in capacity utilization between 2007Q4 and its post-2010 average is -5.42%, which amounts to a parameter value of $\omega = 1.0182$.[18]

In our model, the fixed cost f governs the frequency at which regime transitions occur. In particular, under our interpretation that the US economy stays mostly in the high regime, f determines the probability that the economy can fall in the low regime, which corresponds to a large, persistent fall in GDP. With only thirty years of data, time series averages are only mildly informative about the frequency of these transitions. We propose instead to rely on probabilistic forecasts. More precisely, the SPF provides mean probability forecasts of GDP growth over various bins. According to the survey, the probability that real GDP growth falls below -2%, its lowest category, is on average 0.63%.[19] Adjusting for an average trend growth rate of 2.9% in the SPF

[17] In either the high or the low regime, m is nearly constant close to 1 or 0. Since regime transitions are rare in the US experience, the contribution of m to average output volatility is thus negligible. Using the expression $Y_t = \left((1-\alpha)\frac{\sigma-1}{\sigma}\right)^{\frac{1-\alpha}{\alpha+\nu}}\left(Ae^{\theta_t}\Omega(m_t)K_t^\alpha\right)^{\frac{1+\nu}{\alpha+\nu}}$ with $\Omega(m) = \left(m\left(\omega^{\sigma-1}-1\right)+1\right)^{\frac{1}{\sigma-1}}$, and ignoring the contribution of m, the variance of beliefs about current log output is $\text{Var}\left(\log(Y_t)|\theta_{t-1}, v_t\right) = \left(\frac{1+\nu}{\alpha+\nu}\right)^2\frac{1}{\gamma_\theta+\gamma_v}$ yielding an interquartile range of $\text{IQR} = 2\Phi^{-1}(0.75)\sqrt{\left(\frac{1+\nu}{\alpha+\nu}\right)^2\frac{1}{\gamma_\theta+\gamma_v}}$, where Φ is the CDF of a standard normal. Using the parameters of the calibration together with the SPF data over the period 1985Q1-2015Q2 yields the value of γ_v.

[18] With only two capacity utilization levels, our model cannot explain the sharp, short-lived dip in capacity between 2008 and 2010. Since we are interested in understanding the slow recovery after the recession, we make the conservative choice of removing this period from the post-recession average. Including it would imply a larger ω, widening the gap in output between the high and the low regimes.

[19] We use the mean probability forecast about next year real GDP growth from the SPF. Because the SPF variable definitions change over time, we restrict our sample to 1992Q1-2009Q1, which corresponds to the largest available sample with a consistent definition included in our period of interest.

data, we pick f so that the average probability that output growth in our model will be lower than -4.9% over the next year is consistent with the survey. The calibrated value of f is such that if all firms were to produce at high capacity the fixed costs would amount to about 1% of average output.

The parameters are jointly estimated by a method of simulated moments that minimizes the distance between the empirical and simulated moments, computed over long-run simulations. Table 1 lists the parameters. As it turns out, our resulting parameters are such that condition (18), which guarantees the uniqueness of the global game equilibrium, is satisfied.

Table 1: Parameters

Parameter	Value	Source/Target
Time period	one quarter	
Total factor productivity	$A = 1$	Normalization
Capital share	$\alpha = 0.3$	Labor share 0.7
Discount factor	$\beta = 0.95^{1/4}$	0.95 annual
Depreciation rate	$\delta = 1 - 0.9^{1/4}$	10% annual
Risk aversion	$\gamma = 1$	log utility
Elasticity of labor supply	$\nu = 0.4$	Jaimovich and Rebelo (2009)
Persistence θ process	$\rho_\theta = 0.94$	Autocorrelation of log output
Long-run standard deviation of θ	$\sigma_\theta = 0.009$	Standard deviation of log output
Elasticity of substitution	$\sigma = 3$	Hsieh and Klenow (2014)
Precision of private signal	$\gamma_v = 1,154,750$	See text
TFP gain from high capacity	$\omega = 1.0182$	See text
Fixed cost	$f = 0.021$	See text

4.2 Quantitative Evaluation

The calibrated parameters are such that, because of the coordination problem, the economy has two stable steady-states for intermediate values of θ, but only one steady-state for high or low values of θ.[20]

Ergodic distributions

To illustrate the unusual dynamic properties that result from the steady-state multiplicity, we simulate the model for one million periods and plot the ergodic distributions of measured TFP, output, investment, consumption, employment and the productivity process θ in logs on Figure 8. While productivity θ is normally distributed, the other aggregates are negatively skewed and have bimodal ergodic distributions, a sign that the economy spends a substantial amount of time in the low regime. For consistence with our detrending method, each simulated distribution is centered

[20] Figure 16 in the Appendix shows the dynamics of K for various values of θ.

around the upper mode corresponding to the high regime. For comparison with the data, Figure 9 reproduces the empirical distributions of these variables in log deviation from trend. As the data shows, bimodality is roughly observed for most variables, and our model offers a reasonable fit to the empirical distributions, except for investment which appears more dispersed in the data.

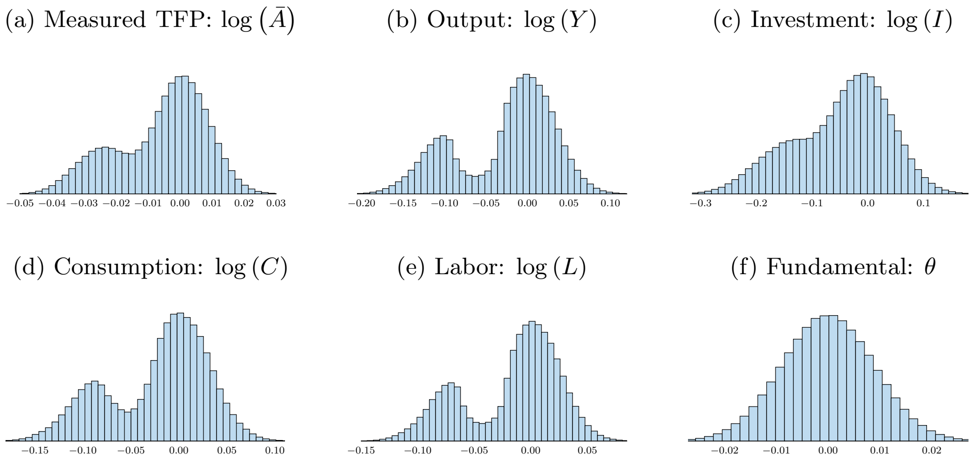


Figure 8: Model: ergodic distributions of simulated data

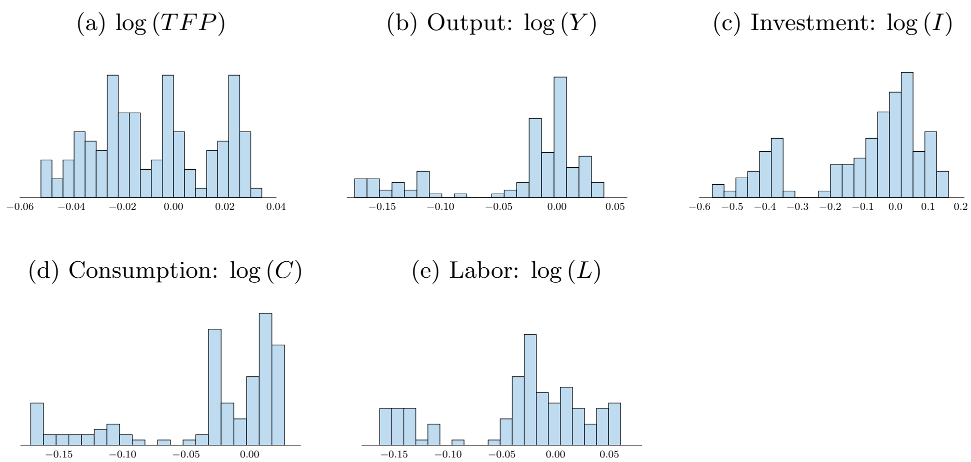


Figure 9: Data: distributions of aggregates

Business cycles moments

To further evaluate the fit of the model, we compute various business cycle moments from simulated time series and compare them to their empirical counterparts. The results are shown in table 2 together with moments generated from a standard real business cycle model.[21] The differences between the full model and the RBC model highlight the influence of the coordination

[21] Without our coordination mechanism to provide amplification and propagation, the aggregate productivity process in the RBC model must be recalibrated in order to fit the autocorrelation and standard deviation of log output.

mechanism on the dynamics of the economy. In terms of standard deviations, and correlations with output, both models perform similarly. Our full model, however, clearly outperforms the RBC model in terms of skewness. This result stems from the presence of the two steady states, which imply that the economy spends a substantial amount of time in the depressed state.[22]

Table 2: Dynamic properties of the data, the full model and the RBC model.

	Output	Investment	Hours	Consumption
	Correlation with output			
Data	1.00	0.90	0.91	0.98
Full model	1.00	0.90	1.00	0.99
RBC model	1.00	0.95	1.00	0.99
	Standard deviation relative to output			
Data	1.00	3.09	1.03	0.94
Full model	1.00	1.44	0.71	0.88
RBC model	1.00	1.30	0.71	0.95
	Skewness			
Data	-1.24	-0.92	-0.62	-1.31
Full model	-0.58	-0.44	-0.58	-0.53
RBC model	-0.00	-0.03	-0.00	-0.00

Impulse response functions

To illustrate the non-linear properties of the model, we now look at the response of various aggregates to productivity shocks. Starting in the high steady state, we hit the economy with three sequences of shocks of different sizes and durations, represented in panel (a) of Figure 10. For the solid blue curve, the innovations in θ are set to -3 standard deviations for 3 quarters; for the dashed red curve the innovations are set to -2.5 standard deviations for 3 quarters. Finally, for the remaining green dotted curve, the innovations are set to -2 standard deviations for 2 quarters. These specific shocks were chosen to illustrate the types of dynamics that the model can generate.

After the small shock, firms reduce their scale of operation only slightly. They keep coordinating on the high capacity throughout the duration of the shock and, as a result, the economy recovers fairly quickly to the high steady state once the shock has disappeared. The response of the economy is essentially the same as what we would observe in a standard RBC model. The situation is different when the economy is hit by the shock of intermediate size, represented by the dashed lines in Figure 10. In this case, firms reduce their production more drastically by changing their capacity utilization rates and cutting down on inputs, partly because of lower productivity and partly because of lower

The long-run standard deviation of θ is recalibrated to $\sigma_\theta = 1.6\%$ and its persistence to $\rho_\theta = 0.97$. Preferences and technology parameters are otherwise the same as in our benchmark calibration.

[22]The negative skewness of these variables is a property of the data robust to changes in the time range. For instance, the skewness of log output over 1967Q1-2015Q2 is -0.47.

aggregate demand. Because of this failure of firms to coordinate on high production, the economy takes substantially more time to recover to the high regime. Finally, after the large shock, capacity utilization drops massively and stays low for a long time. With less resources, the household saves less and the capital stock declines, making coordination on the high capacity even more difficult. The economy converges to the low steady state and remains trapped there even after productivity θ has fully recovered. Once in the low regime, only a sufficiently large positive shock can move the economy back into the basin of attraction of the high steady state.[23]

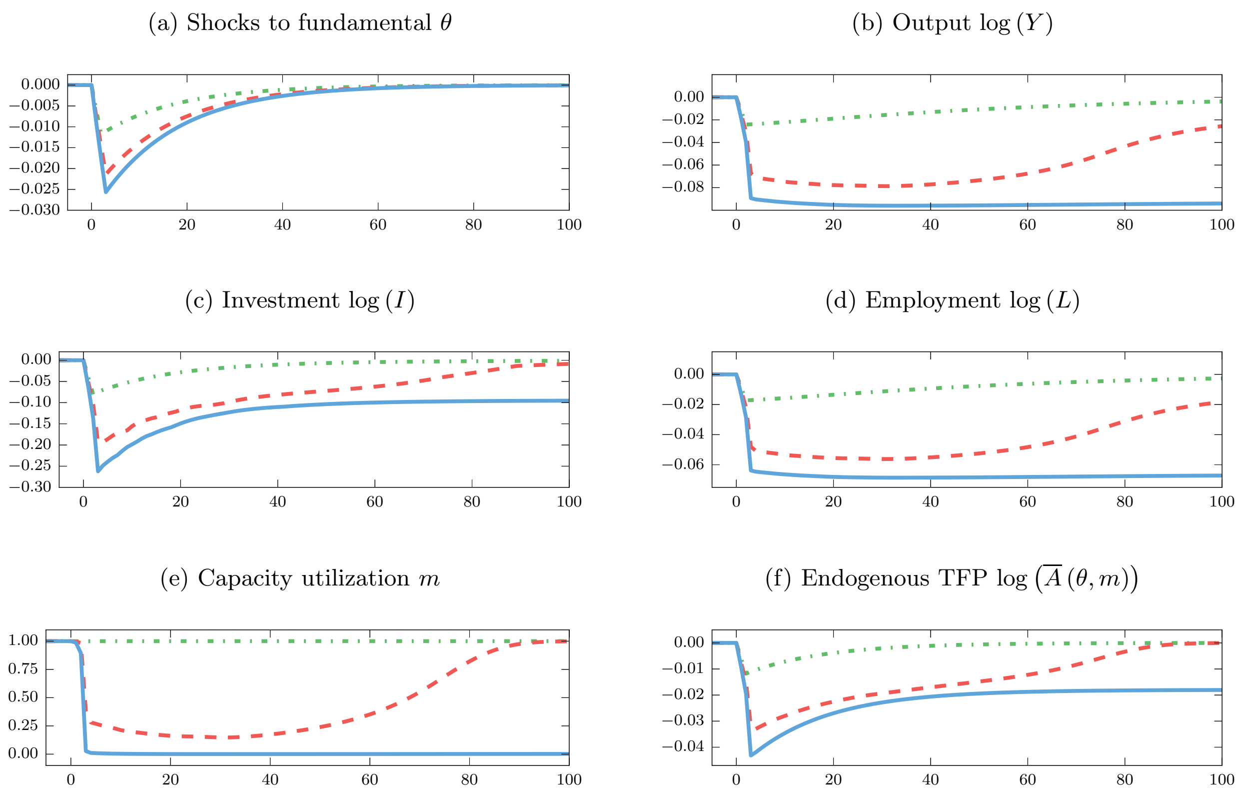


Figure 10: Impulse response functions

[23]Figure 17 in the Appendix plots the response of the return on capital $R-\delta$ and the wage W to these shocks. Both of them drop on impact. When the economy settles in the low steady state, wages remain depressed, while the return on capital recovers to its initial long-run value. Note that R corresponds to the real rental rate of productive capital and is therefore the return on a risky asset. The behavior of R in our model is consistent with the the detrended yield on AAA and BAA Corporate Bonds corrected for inflation expectations, which had recovered by 2014.

The aftermath of the 2007-2009 recession

We now turn our attention to the Great Recession. Panel (a) of Figure 11 shows the behavior of output, employment, investment, consumption and TFP[24] from 2005 to the second quarter of 2015. All series are normalized to 0 at the beginning of the recession in 2007Q4. After the initial hit, consumption, output and employment slowly declined and stabilized at about 10% below their pre-recession levels.[25] Similarly, investment initially dropped by about 45% before recovering to 25% below its pre-recession level.

To evaluate whether our model can replicate the US experience during this recession, we reverse-engineer a series of productivity shocks θ so that the endogenous TFP in our model matches the measured TFP series between 2007Q4 and 2009Q2. The economy starts from the high steady state corresponding to $\theta = 0$. We set the innovations to productivity to zero after 2009Q3 and let the economy recover afterwards. As it turns out, such a series of shocks is enough to trigger a shift to the low regime.[26] The response of various aggregates is shown in panel (b) of Figure 11. As we can see, our model offers a reasonable description for the evolution of consumption, employment and output. Notice also that our model provides an endogenous explanation for the protracted decline in measured TFP.[27] The reaction of investment, on the other hand, is more muted in our model compared to the data as it falls by 32% on impact and then stabilizes at about 15% below its initial trend.[28] In the simulation, the initial drop in endogenous TFP is due to the direct impact of the productivity shock together with the transition from high to low capacity by the firms. Its long-run behavior, however, is solely driven by the low capacity, as exogenous productivity θ has completely recovered by then.

[24]TFP is measured as the Solow residual and is analogous to the endogenous TFP $\overline{A}(\theta, m)$ from the model. See Appendix B for details.

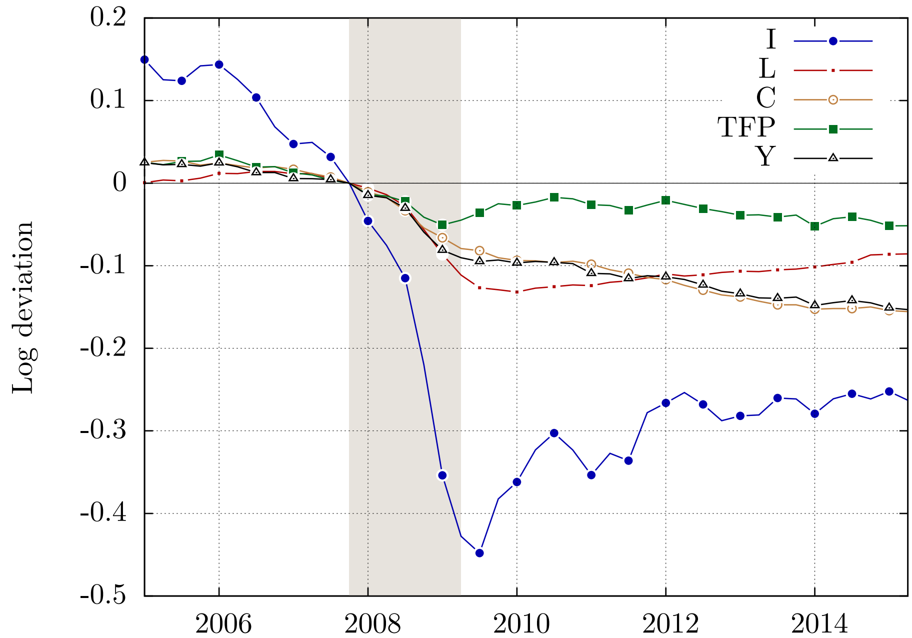
[25]Note also that the average growth rate of output after the recession is slightly lower than its average over the 1985Q1-2007Q3 period, so that the economy is moving away from its earlier trend. While our current theory cannot explain it, a modified version with investment in R&D could potentially address this fact. See Benigno and Fornaro (2015) for a similar explanation.

[26]Our counterfactual experiment relies on aggregate productivity shocks only as our objective is not to provide a complete story for the 2007-2009 episode but for the recovery period that followed. As we mentioned earlier, our coordination mechanism may provide equally strong propagation to other theories of the recession based on financial shocks, policy changes, uncertainty and others.

[27]The fact that our explanation relies on an endogenous drop in TFP rather than the usual exogenous one has important implications for the role of policy which we explore in the next section.

[28]In the data, a substantial fraction of the drop in investment is due to a decline in residential investment, which the model does not address.

(a) Data: US series centered on 2007Q4



(b) Model

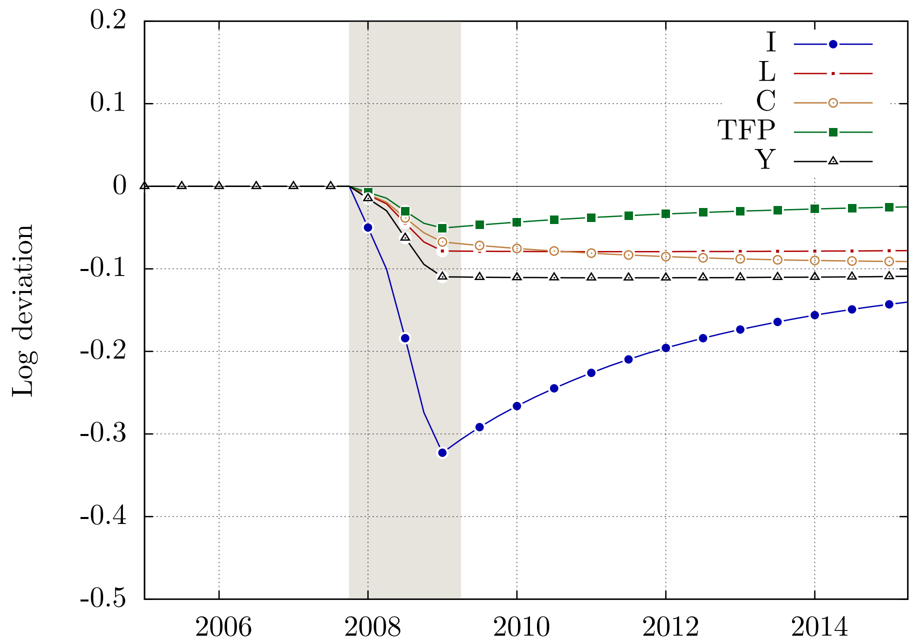


Figure 11: The 2007-2009 recession and its aftermath

5 Policy

The prospect of coordination failures is often used in policy debates to justify large government interventions, including in particular expansionary fiscal policies. In this section, we study the appropriate policy response in our model when the economy is hindered by a coordination problem and discuss to what extent policies such as government spending may be beneficial, if at all desirable. We first solve for the efficient allocation and describe how it can be implemented using various subsidies. We then consider whether an increase in government spending can be welfare improving when the efficient subsidies are not available.

5.1 Efficient Allocation

Our model economy suffers from two related inefficiencies. The first inefficiency arises as firms use their monopoly power to price their products at a markup over their marginal cost. As a result, firms produce and sell too little. The second inefficiency is due to the effect of the aggregate demand externality on capacity decisions. Firms do not internalize that producing at high capacity positively impacts the demand that other producers face and therefore fail to coordinate on the efficient capacity level.

To shed light on these inefficiencies, we solve the problem of a constrained social planner that does not receive any signal and cannot aggregate the information available to private agents as in Angeletos and Pavan (2007). The planner can, however, instruct each firm to use the high capacity with some probability $z(v) \in [0,1]$ as a function of its private signal v. With this policy instrument, the planner's problem is

$$V_{SP}(K,\theta_{-1}) = \max_{0\leqslant z(\cdot)\leqslant 1} \mathbb{E}\left[\max_{K',L} U\left(\bar{A}\left(\theta,m\right)K^{\alpha}L^{1-\alpha} - m\left(\theta,z\right)f - K' + (1-\delta)K, L\right) \right.$$
$$\left. + \beta V_{SP}(K',\theta)\middle|\theta_{-1}\right]$$

where $m\left(\theta,z\right) = \int \sqrt{\gamma_v}\phi\left(\sqrt{\gamma_v}\left(v-\theta\right)\right)z(v)dv$. Notice that we already use the result, shown in the proof in Appendix F.5, that the economy admits aggregation and directly write the planner's problem using the aggregate production function.

We characterize the constrained efficient allocation and its implementation in the following proposition.

Proposition 5. *The competitive equilibrium with incomplete information is inefficient, but the constrained efficient allocation can be implemented with a lump-sum tax on the household, an input subsidy s_{kl} and a profit subsidy s_π to intermediate goods producers such that $1 - s_{kl} = \frac{\sigma-1}{\sigma}$ and $1 + s_\pi = \frac{\sigma}{\sigma-1}$.*

Proposition 5 shows that the constrained efficient allocation can be implemented in the competitive economy using simple instruments that correct the two distorted margins directly. To offset the distortions induced by the monopoly power, the planner uses an input subsidy s_{kl}, standard in the New-Keynesian literature, to encourage firms to expand to the optimal scale of operation. Despite this input subsidy, firms still operate at a suboptimal capacity level because of the aggregate demand externality and the planner needs an additional instrument to induce the right capacity choice. Perhaps surprisingly, a simple linear profit subsidy s_π is enough to correct this margin in the global game. By increasing profits, this subsidy makes firms internalize the impact of their capacity choice on aggregate demand and incentivizes the adoption of the high capacity. As a result, one should expect firms to coordinate more easily on high capacity under the optimal policy, and the basin of attraction for the high regime should consequently expand. In other words, the economy would visit the low regime less frequently and the incidence and persistence of deep recessions would be reduced. Finally, to complete the implementation, we use a non-distortionary lump-sum tax on the household to ensure that the government budget constraint balances every period.[29]

5.2 Government Spending

The optimal implementation result involves the use of input and profit subsidies. In the event that such instruments are unavailable to policymakers, for instance due to political economy reasons, we consider the impact of government spending on the economy. Since firms operate at an inefficiently low capacity level in equilibrium, an increase in aggregate demand caused by government spending may, in principle, have a positive impact on welfare by raising the incentives to adopt the high capacity. We investigate this claim in the context of our model.

We find that, in general, government spending is detrimental to welfare because the crowding out of private investment hurts coordination in subsequent periods. Government spending thus creates dynamic welfare losses. However, we also find that government spending can be welfare improving in a small region of the state space if the preferences of the household allow for a wealth effect on the labor supply.

We now describe how these two channels operate. To do so, we assume that government spending is pure government consumption not valued by the household and financed through a lump-sum tax on the household.[30]

Crowding out of private spending

As in the neoclassical growth model, an increase in government spending leads to a reduction in the wealth of households which, as a result, save less in physical capital. Consequently, the amount of capital available in the following periods is reduced, which hurts coordination and reduces

[29]This implementation is not unique and we show, in Appendix F.5, that another implementation based on a single sales subsidy can correct both margins at the same time because of the specific structure implied by the Dixit-Stiglitz model of monopolistic competition.

[30]As Ricardian equivalence holds in our environment the timing of taxes is irrelevant.

the measure of firms adopting the high capacity, in contrast to what efficiency requires. In this sense, perhaps in contradiction with the common intuition, the coordination problem magnifies the crowding out effect of government spending in our model.

We can precisely establish this point in our benchmark framework. With GHH preferences, the crowding out effect associated with government spending unambiguously leads to welfare losses.

Proposition 6. *Under GHH preferences, for γ_v large, an unforeseen one-time increase in government spending financed by lump-sum taxes reduces welfare.*

The intuition behind this result is as follows. Under GHH preferences, the equilibrium output and employment only depend on current capital K, productivity θ and the measure of firms with high capacity m. When γ_v is large, risk at the time of the capacity choice is negligible and the stochastic discount factor is irrelevant in the surplus expression (16). As a result, in the limit as $\gamma_v \to \infty$, government spending has no impact on the outcome of the current coordination game. The measure m remains unaffected and only the crowding out effect remains. Government spending is thus a pure waste of resources.

Wealth effect on the labor supply

When the assumption of GHH preferences is relaxed, the labor supply curve of the household is affected by government spending.[31] As the household gets poorer, labor supply goes up, thereby putting downward pressure on wages. With cheaper inputs, firms expand and are more tempted to use the high capacity, which alleviates the coordination problem and may result in welfare gains.

Figure 12 illustrates the mechanism. The upper (red) and the lower (blue) curves represent the high and the low equilibria of the model with complete information. The black curves represent the unique equilibrium of the model with incomplete information, with and without government spending G. As government spending increases, firms are more tempted to use the high capacity and the zone with multiple equilibria shifts to the left, from the dotted to the shaded region. As a result, the low equilibrium ceases to exist for the range of K to the right of the shaded region: the wage would be so low in that equilibrium that operating at a large scale with high capacity would always be preferable. In the environment with incomplete information, the equilibrium of the global game lies between the two equilibria of the complete information setup. The curve that indicates the unique equilibrium selected by the global game therefore also shifts to the left, from the dashed curve to the solid one. Notice that for values of K in the transition region, the resulting increase in the mass of firms using the high capacity increases the endogenous TFP $\bar{A}$ which increases output, and, potentially, consumption, investment and welfare. Additionally, government spending can also helps coordination in subsequent periods. If it succeeds in raising investment, government spending can move the economy from the bad regime to the good one, therefore generating potentially large dynamic welfare gains.

[31] We can no longer derive all our theoretical results under these new preferences, but the model can be solved numerically. We make sure, in particular, that uniqueness still obtains for the global game in our numerical simulations.

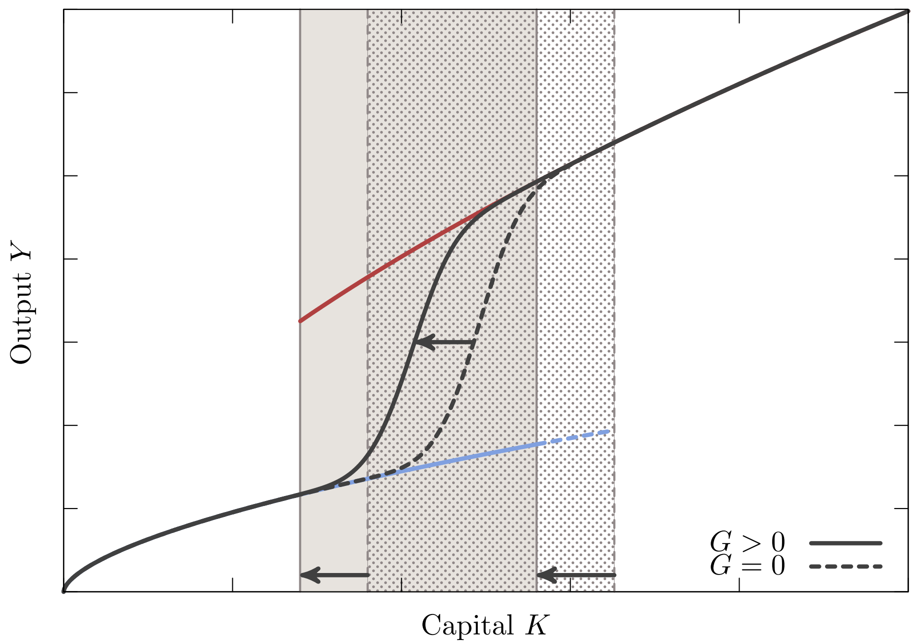


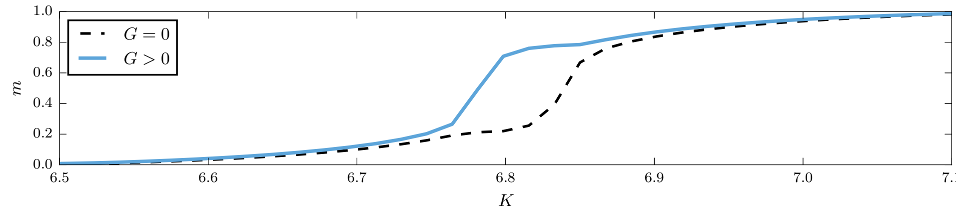
Figure 12: Impact of an increase in government spending on coordination

Numerical simulations

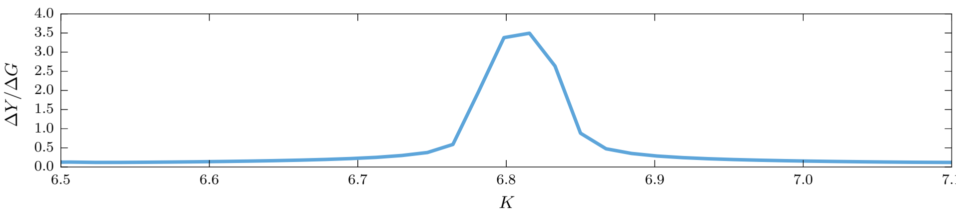
To illustrate the overall impact of government spending on the economy, we proceed to a series of simulations. To allow for a wealth effect on the labor supply, we relax the assumption of GHH preferences and use instead standard separable preferences $U(C, L) = \log C - (1+\nu)^{-1} L^{1+\nu}$. The parameters and the details of the exercise are included in Appendix D. We consider an economy in which government spending G_t is high $G_t = G > 0$ with probability 1/2 and low $G_t = 0$ with probability 1/2. The draws are independent across time. We set G to equal 0.5% of the steady-state level of output and we assume that the value of G is revealed to all agents at the beginning of the period.

Figure 13 shows the outcome of these simulations. In the top panel, we see that an increase in government spending G helps firms coordinate on the high capacity in some region of the state space. Interestingly, this effect is only present for values of K in which the economy is close to the transition in m between the low and the high regime. Elsewhere, G has little to no impact on coordination. On Panel (b), we see that the interaction of coordination and government spending can give rise to large contemporaneous multipliers for output. When the gains from coordination are large enough to offset the crowding out effect, government spending may improve welfare, as expressed in consumption equivalent terms on panel (c). Notice, however, that government spending is generally detrimental to welfare, as the dynamic welfare losses coming from the crowding out effect dominate in most of the state space. Only in the region where the economy is close to a transition from the low to the high regime does government spending help coordination sufficiently to improve welfare. This result highlight the importance of the timing of government intervention.[32]

(a) Impact of G on m



(b) Output multiplier



(c) Impact of G on welfare

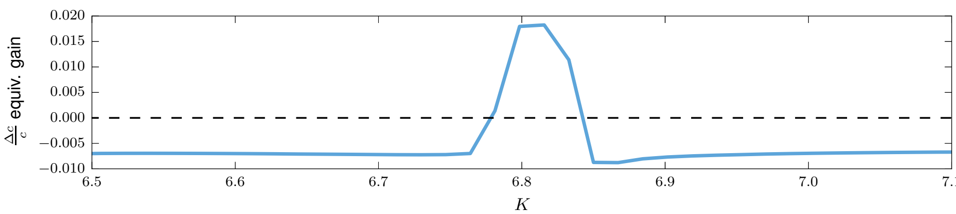


Figure 13: Impact of an increase in government spending for $\theta = \theta_{-1} = 0$

6 Conclusion

We develop a dynamic stochastic general equilibrium model of business cycles with coordination failures. The model provides an alternative foundation for Keynesian-type demand-deficient downturns as the economy may fall into long-lasting recessions due to the failure of firms to coordinate on a higher output. The calibrated model outperforms the RBC benchmark in terms of business cycles asymmetries. It also replicated salient features of the slow recovery from the Great Recession. Government spending policies are generally detrimental to welfare, but may sometimes be welfare improving, without relying on nominal rigidities, when the economy is about to transition between regimes.

[32]In a recent paper covering the period 1947Q1 to 2008Q4, Auerbach and Gorodnichenko (2012) find that the fiscal multiplier for total government spending in the United States is in general small but exceeds 1 during recessions. This evidence is potentially consistent with the model if the small recessions during that period coincide with episodes during which the economy slightly enters the transition zone before recovering.

In this paper, we have limited the scope of our policy analysis to simple subsidies and a basic government spending policy, but other types of interventions may help alleviate the coordination problem. For instance, in the presence of nominal rigidities, monetary policy may encourage coordination in the future by affecting interest rates and the rate of accumulation of capital. Investment subsidies and other types of government spending may also lead to different conclusions.

Non-convexities in the firm's problem are an essential part of our mechanism. In this paper, we have focused on a simple binary capacity utilization choice, but we believe that the central mechanism of the paper applies to a larger class of non-convexities. For instance, it would be interesting to extend the model to include fixed cost of adjusting capital or labor, which have been widely documented in the empirical literature.

More broadly, we believe that the interaction of non-convexities and complementarities can generate interesting mechanisms in other contexts. For instance, the possibility of falling in the low regime may have interesting asset pricing implications, as we can interpret our model as providing a theory of endogenous rare disasters. Another likely important factor influencing coordination is social learning. In Schaal and Taschereau-Dumouchel (2015), we consider an environment in which people learn from the actions of others in a coordination game. The interaction of complementaries and social learning give rise to exuberant periods of economic activity followed by brutal crashes.

References

Angeletos, G.-M., C. Hellwig, and A. Pavan (2007): "Dynamic global games of regime change: Learning, multiplicity, and the timing of attacks," *Econometrica*, 75, 711–756.

Angeletos, G.-M. and J. La'O (2013): "Sentiments," *Econometrica*, 81, 739–779.

Angeletos, G.-M. and A. Pavan (2007): "Efficient Use of Information and Social Value of Information," *Econometrica*, 75, pp. 1103–1142.

Angeletos, G.-M. and I. Werning (2006): "Crises and prices: Information aggregation, multiplicity, and volatility," *The American economic review*, 96, 1720–1736.

Atkeson, A. (2000): "Rethinking Multiple Equilibria in Macroeconomic Modeling: Comment," *NBER Macroeconomics Annual*, 15, pp. 162–171.

Auerbach, A. J. and Y. Gorodnichenko (2012): "Measuring the Output Responses to Fiscal Policy," *American Economic Journal: Economic Policy*, 4, 1–27.

Azariadis, C. and A. Drazen (1990): "Threshold externalities in economic development," *The Quarterly Journal of Economics*, 501–526.

Benhabib, J. and R. E. Farmer (1994): "Indeterminacy and Increasing Returns," *Journal of Economic Theory*, 63, 19 – 41.

Benhabib, J., P. Wang, and Y. Wen (2015): "Sentiments and Aggregate Demand Fluctuations," *Econometrica*, 83, 549–585.

Benhabib, J. and Y. Wen (2004): "Indeterminacy, aggregate demand, and the real business cycle," *Journal of Monetary Economics*, 51, 503 – 530.

Benigno, G. and L. Fornaro (2015): "Stagnation Traps," *manuscript*.

Bernard, A. B., J. Eaton, J. B. Jensen, and S. Kortum (2003): "Plants and Productivity in International Trade," *The American Economic Review*, 93, pp. 1268–1290.

Bresnahan, T. F. and V. A. Ramey (1994): "Output Fluctuations at the Plant Level," *The Quarterly Journal of Economics*, 593–624.

Broda, C. and D. E. Weinstein (2006): "Globalization and the Gains From Variety," *The Quarterly Journal of Economics*, 121, 541–585.

Burdzy, K., D. M. Frankel, and A. Pauzner (2001): "Fast equilibrium selection by rational players living in a changing world," *Econometrica*, 69, 163–189.

Caballero, R., E. Engel, and J. Haltiwanger (1997): "Aggregate Employment Dynamics: Building from Microeconomic Evidence," *The American Economic Review*, 87, 115–137.

Carlsson, H. and E. Van Damme (1993): "Global Games and Equilibrium Selection," *Econometrica*, 61, pp. 989–1018.

Chamley, C. (1999): "Coordinating regime switches," *The Quarterly Journal of Economics*, 114, 869–905.

Christiano, L. J., M. S. Eichenbaum, and M. Trabandt (2015): "Understanding the Great Recession," *American Economic Journal: Macroeconomics*, 7, 110–67.

Coleman, I. and W. John (2000): "Uniqueness of an equilibrium in infinite-horizon economies subject to taxes and externalities," *Journal of Economic Theory*, 95, 71–78.

Coleman, W. J. (1991): "Equilibrium in a production economy with an income tax," *Econometrica: Journal of the Econometric Society*, 1091–1104.

Cooper, R. and A. John (1988): "Coordinating Coordination Failures in Keynesian Models," *The Quarterly Journal of Economics*, 103, pp. 441–463.

Cooper, R. W. and J. C. Haltiwanger (2006): "On the nature of capital adjustment costs," *The Review of Economic Studies*, 73, 611–633.

Datta, M., L. J. Mirman, and K. L. Reffett (2002): "Existence and uniqueness of equilibrium in distorted dynamic economies with capital and labor," *Journal of Economic Theory*, 103, 377–410.

Diamond, P. A. (1982): "Aggregate demand management in search equilibrium," *The Journal of Political Economy*, 881–894.

Doms, M. and T. Dunne (1998): "Capital adjustment patterns in manufacturing plants," *Review of Economic Dynamics*, 1, 409–429.

Fajgelbaum, P., E. Schaal, and M. Taschereau-Dumouchel (2015): "Uncertainty Traps," *manuscript*.

Farmer, R. E. and J.-T. Guo (1994): "Real business cycles and the animal spirits hypothesis," *Journal of Economic Theory*, 63, 42–72.

Feenstra, R. C., R. Inklaar, and M. Timmer (2015): "The next generation of the Penn World Table," Tech. rep.

Fernald, J. G. (2014): "A Quarterly, Utilization-Adjusted Series on Total Factor Productivity," *Manuscript, Federal Reserve Bank of San Francisco.*

Frankel, D. and A. Pauzner (2000): "Resolving indeterminacy in dynamic settings: the role of shocks," *The Quarterly Journal of Economics*, 115, 285–304.

Gertler, M. and P. Karadi (2011): "A model of unconventional monetary policy," *Journal of monetary Economics*, 58, 17–34.

Goldstein, I. and A. Pauzner (2005): "Demand-Deposit Contracts and the Probability of Bank Runs," *The Journal of Finance*, 60, pp. 1293–1327.

Gourio, F. (2012): "Disaster Risk and Business Cycles," *The American Economic Review*, 102, 2734–2766.

Greenwood, J., Z. Hercowitz, and G. W. Huffman (1988): "Investment, Capacity Utilization, and the Real Business Cycle," *The American Economic Review*, 78, pp. 402–417.

Guimaraes, B. and C. Machado (2014): "Demand Expectations and the Timing of Stimulus Policies," *Manuscript*.

Hall, G. J. (2000): "Non-convex costs and capital utilization: A study of production scheduling at automobile assembly plants," *Journal of Monetary Economics*, 45, 681–716.

Hall, R. E. (2014): "Quantifying the Lasting Harm to the US Economy from the Financial Crisis," in *NBER Macroeconomics Annual 2014, Volume 29*, University of Chicago Press.

Hansen, G. D. and E. C. Prescott (2005): "Capacity constraints, asymmetries, and the business cycle," *Review of Economic Dynamics*, 8, 850 – 865.

Hsieh, C.-T. and P. J. Klenow (2014): "The Life Cycle of Plants in India and Mexico," *The Quarterly Journal of Economics*, 129, 1035–1084.

Jaimovich, N. and S. Rebelo (2009): "Can News about the Future Drive the Business Cycle?" *American Economic Review*, 99, 1097–1118.

Jones, L. E. and R. E. Manuelli (1992): "The Coordination Problem and Equilibrium Theories of Recessions," *The American Economic Review*, 82, pp. 451–471.

Kiyotaki, N. (1988): "Multiple Expectational Equilibria under Monopolistic Competition," *The Quarterly Journal of Economics*, 103, pp. 695–713.

Morand, O. F. and K. L. Reffett (2003): "Existence and uniqueness of equilibrium in nonoptimal unbounded infinite horizon economies," *Journal of Monetary Economics*, 50, 1351–1373.

Morris, S. and H. S. Shin (1998): "Unique Equilibrium in a Model of Self-Fulfilling Currency Attacks," *The American Economic Review*, 88, pp. 587–597.

——— (1999): "A theory of the onset of currency attacks," *The Asian Financial Crisis: Causes, Contagion and Consequences*, 2, 230.

——— (2003): "Global Games: Theory and Applications," *Advances in Economics and Econometrics*, 56.

Murphy, K. M., A. Shleifer, and R. Vishny (1989): "Industrialization and the Big Push," *The Journal of Political Economy*, 97, 1003–1026.

Ramey, V. A. (1991): "Nonconvex costs and the behavior of inventories," *Journal of Political Economy*, 306–334.

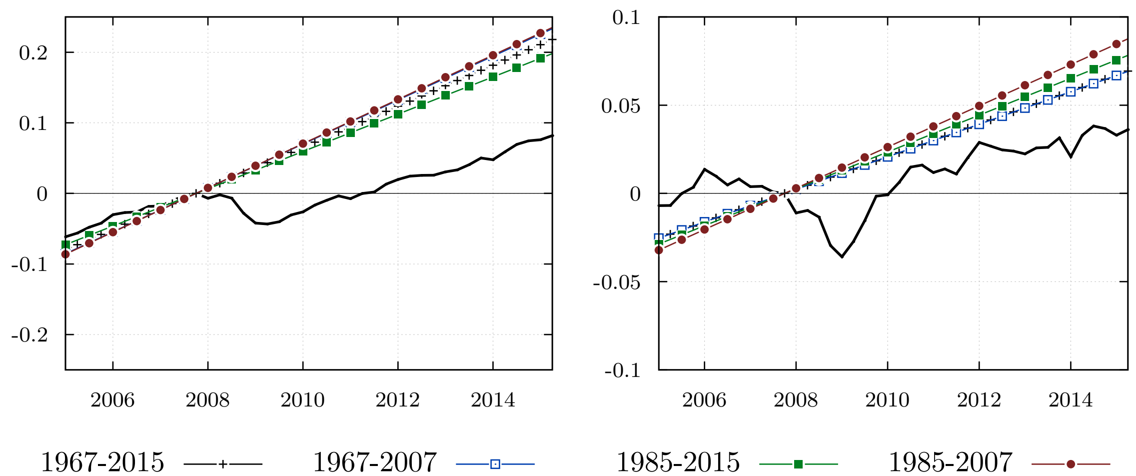
Schaal, E. and M. Taschereau-Dumouchel (2015): "Learning to Coordinate over the Business Cycle," *manuscript*.

Tarski, A. et al. (1955): "A lattice-theoretical fixpoint theorem and its applications," *Pacific journal of Mathematics*, 5, 285–309.

Wen, Y. (1998): "Capacity Utilization under Increasing Returns to Scale," *Journal of Economic Theory*, 81, 7 – 36.

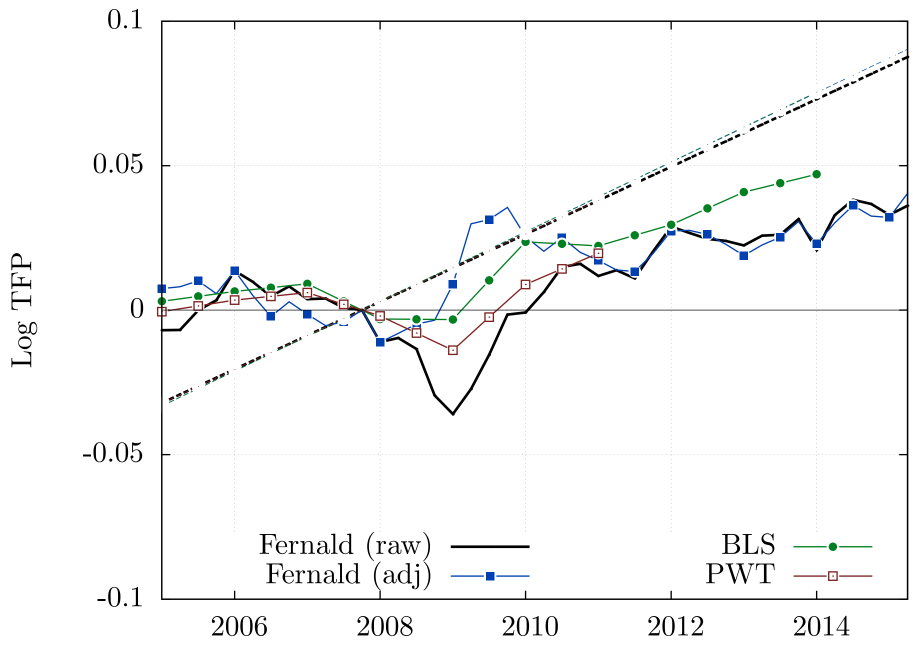
A Additional Figures - For online publication

(a) log RGDP (b) log TFP



Notes: Data in logs, undetrended, all series centered at 2007Q4, including trends. Linear trends: 1967Q1-2015Q2, 1967Q1-2007Q3, 1985Q1-2007Q3 and 1985Q1-2015Q2. TFP is the raw TFP measure from Fernald (2014).

Figure 14: Impact of detrending on GDP and TFP



Notes: Data in logs, undetrended, all series centered at 2007Q4. The linear trends are computed over the intersection of period 1985Q1-2015Q2 with the series availability. These trends coincide almost exactly for the four series. Fernald (raw) is the raw TFP measure from Fernald (2014), adjusted for labor quality. Fernald (adj.) is the same measure adjusted for capacity utilization. BLS is the BLS multifactor productivity for the total private sector. PWT is the Penn World Tables measure of TFP, see Feenstra et al. (2015), data available at www.ggdc.net/pwt.

Figure 15: Various measures of TFP over 2005-2015

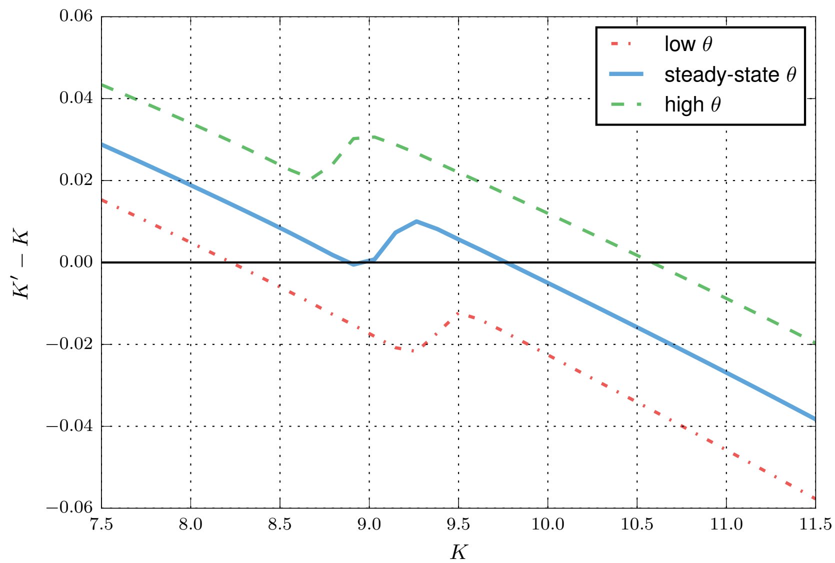
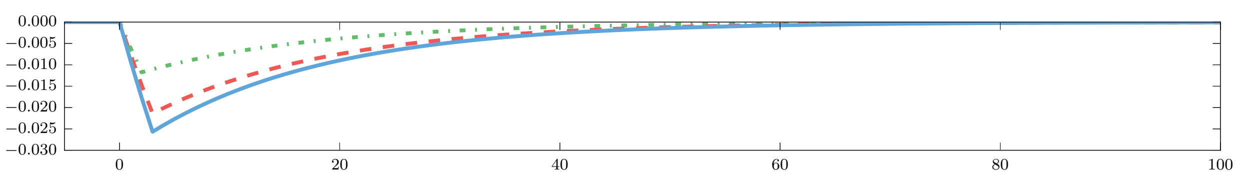
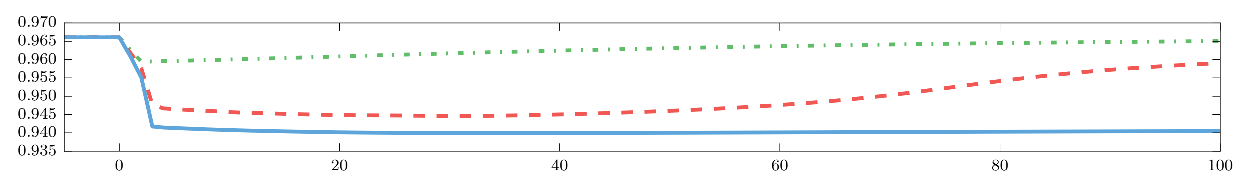


Figure 16: Dynamics of capital K for different values of θ.

(a) Shocks to productivity θ



(b) Wage W



(c) Return on capital $R - \delta$

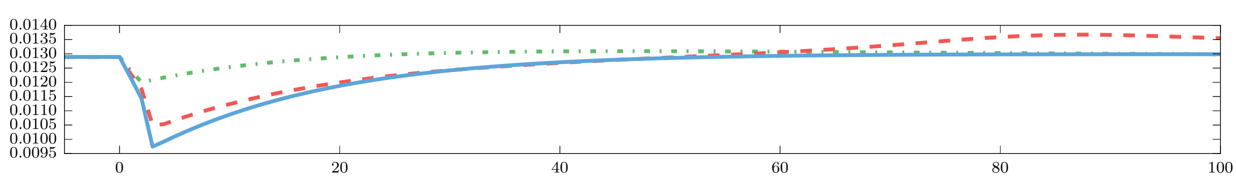


Figure 17: Impulse response functions for prices

B Data - For online publication

Table 3 details the data sources. All time series are quarterly from 1985Q1 to 2015Q2.[33] All time series except capacity utilization are seasonally adjusted. For all time series except capacity utilization, we remove a linear trend from the log series.

Table 3: Data sources

Variable	Source
Output	BEA - Real Gross Domestic Product
Investment	BEA - Real Gross Domestic Investment
Hours	BLS - Nonfarm Business sector: Hours of all persons
Consumption	BEA - Real Personal Consumption Expenditures
Capacity utilization	FRB - Capacity Utilization: Total Industry
Total Factor Productivity	Fernald (2014): Raw Business Sector TFP

C Calibration with $\sigma = 5$ - For online publication

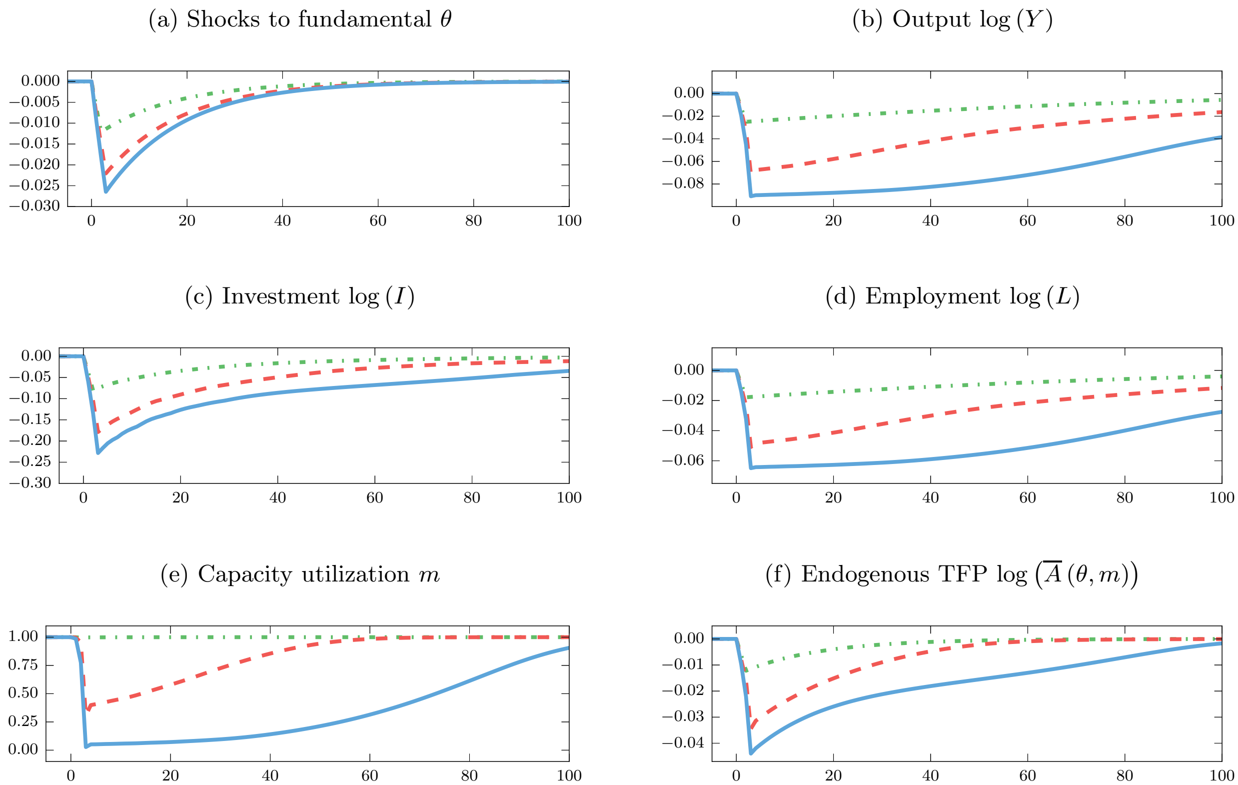
This section contains a full calibration of the model with an elasticity of substitution of $\sigma = 5$. We calibrate this model exactly as in section 4. The parameters are the same as in the benchmark simulation except for those included in table 4. Since the complementarity is weaker, the fixed cost must be higher to match the calibration targets. If all firms produce at high capacity, 1.4% of average output is used to pay the fixed costs.

Parameter	Value	Source/Target
Elasticity of substitution	$\sigma = 5$	
Fixed cost	$f = 0.0535$	See text of section 4.1
Precision of private signal	$\gamma_v = 1,161,250$	See text of section 4.1

Table 4: Calibration with $\sigma = 5$. Parameters that differ from the benchmark calibration of table 1.

This economy does not feature multiple steady states in the dynamics of capital for $\theta_{-1} = \theta = 0$. As a result, large shocks do not lead to permanent downturns but the mechanism still increases the duration of recessions. Figure 18 shows how the economy responds to the same shocks as those used in figure 10. Unlike in the economy with $\sigma = 3$, the economy eventually recovers from the large shock but the coordination mechanism slows down the recovery substantially.

[33] Before 1985, the economy appears to be on a different trend. We therefore limit ourselves to data after 1985 to simplify the detrending.



Notes: For the small shock, the innovation of θ is set to -2 standard deviations for 2 periods. For the medium shock, the innovation of θ is set to -2.5 standard deviations for 3 periods. For the large shock, the innovation of θ is set to -3 standard deviations for 3 periods.

Figure 18: Impulse response functions in the calibrated economy with $\sigma = 5$

D Simulation of Government Spending - For online publication

Table 5 details the parameters of the model used to illustrate the impact of shocks to government spending. We take the same parameters as the calibration except when necessary to highlight the mechanism.

Table 5: Parameters

Parameter	Value	Source/Target
Time period	one quarter	
Total factor productivity	$A = 1$	Normalization
Capital share	$\alpha = 0.3$	Labor share 0.7
Discount factor	$\beta = 0.95^{1/4}$	0.95 annual
Depreciation rate	$\delta = 1 - 0.9^{1/4}$	10% annual
Risk aversion	$\gamma = 1$	log utility
Elasticity of labor supply	$\nu = 0.4$	Jaimovich and Rebelo (2009)
Persistence θ process	$\rho_\theta = 0.94$	Autocorrelation of log output
Long-run standard deviation of ϵ_θ	$\sigma_\theta = 0.006$	Standard deviation of log output
Elasticity of substitution	$\sigma = 3$	Hsieh and Klenow (2014)
Fixed cost	$f = 0.016$	See text
TFP gain from high capacity	$\omega = 1.0182$	See text
Precision of private signal	$\gamma_v = 1{,}013{,}750$	See text
Size of government spending	$G = 0.00662$	0.5% of average output

E Complete Information: Proofs - For online publication

E.1 Equilibrium characterization

Proposition 1. *For a given measure m_t of firms with high capacity the equilibrium output of the final good is given by*

$$Y_t = \overline{A}\left(\theta_t, m_t\right) K_t^{\alpha} L_t^{1-\alpha},$$

where $\overline{A}\left(\theta_t, m_t\right) = \left(m_t A_h\left(\theta_t\right)^{\sigma-1} + \left(1-m_t\right) A_l\left(\theta_t\right)^{\sigma-1}\right)^{\frac{1}{\sigma-1}}$ *and aggregate labor is*

$$L_t = \left[\left(1-\alpha\right)\frac{\sigma-1}{\sigma}\overline{A}\left(\theta_t, m_t\right) K_t^{\alpha}\right]^{\frac{1}{\alpha+\nu}}.$$

The corresponding production and profit levels of intermediate firms are, for $i \in \{h, l\}$,

$$Y_{it} = \left(\frac{A_i\left(\theta_t\right)}{\overline{A}\left(\theta_t, m_t\right)}\right)^{\sigma} Y_t \text{ and } \Pi_{it} = \frac{1}{\sigma}\left(\frac{A_i\left(\theta_t\right)}{\overline{A}\left(\theta_t, m_t\right)}\right)^{\sigma-1} Y_t.$$

Proof. The household's problem delivers the two standard conditions

$$U_c\left(C_t, L_t\right) = \beta\mathbb{E}\left[\left(R_{t+1} + 1 - \delta\right) U_c\left(C_{t+1}, L_{t+1}\right)\right] \text{ and } L_t^{\nu} = \frac{W_t}{P_t}. \tag{19}$$

The first order conditions for an individual firm of type $i \in \{h, l\}$ in terms of capital and labor are

$$\alpha\frac{\sigma-1}{\sigma}\frac{P_t Y_t^{\frac{1}{\sigma}} Y_{it}^{1-\frac{1}{\sigma}}}{K_{it}} = R_t \text{ and } \left(1-\alpha\right)\frac{\sigma-1}{\sigma}\frac{P_t Y_t^{\frac{1}{\sigma}} Y_{it}^{1-\frac{1}{\sigma}}}{L_{it}} = W_t. \tag{20}$$

Combining both equations, we obtain the expression

$$\frac{\sigma-1}{\sigma}Y_t^{\frac{1}{\sigma}}Y_{it}^{-\frac{1}{\sigma}}=\frac{1}{A_i\left(\theta_t\right)}\left(\frac{R_t}{\alpha}\right)^{\alpha}\left(\frac{W_t}{1-\alpha}\right)^{1-\alpha}.$$

Since $\frac{P_{it}}{P_t}=\left(\frac{Y_{it}}{Y_t}\right)^{-\frac{1}{\sigma}}$, we recognize in this expression the optimal strategy for firms to price their products at a constant markup $\frac{\sigma}{\sigma-1}$ over marginal cost (recall that $P_t=1$),

$$P_{it}=\frac{\sigma}{\sigma-1}\frac{1}{A_i\left(\theta_t\right)}\left(\frac{R_t}{\alpha}\right)^{\alpha}\left(\frac{W_t}{1-\alpha}\right)^{1-\alpha}. \tag{21}$$

The price of the final good is

$$P_t=\left(m_tP_{ht}^{1-\sigma}+\left(1-m_t\right)P_{lt}^{1-\sigma}\right)^{\frac{1}{1-\sigma}}=\frac{\sigma}{\sigma-1}\frac{1}{\overline{A}\left(\theta_t,m_t\right)}\left(\frac{R_t}{\alpha}\right)^{\alpha}\left(\frac{W_t}{1-\alpha}\right)^{1-\alpha}. \tag{22}$$

We may then express individual production

$$Y_{it}=\left(\frac{P_{it}}{P_t}\right)^{-\sigma}Y_t=\left(\frac{A_i\left(\theta_t\right)}{\overline{A}\left(\theta_t,m_t\right)}\right)^{\sigma}Y_t, \tag{23}$$

and aggregate output of the final good

$$Y_t=\left(m_tY_{ht}^{\frac{\sigma-1}{\sigma}}+\left(1-m_t\right)Y_{lt}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}=\overline{A}\left(\theta_t,m_t\right)K_t^{\alpha}L_t^{1-\alpha}. \tag{24}$$

Derive the factor demands from the first order conditions,

$$K_{it}=\alpha\frac{\sigma-1}{\sigma}\frac{P_tY_t^{\frac{1}{\sigma}}Y_{it}^{1-\frac{1}{\sigma}}}{R_t}=\alpha\frac{\sigma-1}{\sigma}\left(\frac{A_i\left(\theta_t\right)}{\overline{A}\left(\theta_t,m_t\right)}\right)^{\sigma-1}\frac{P_tY_t}{R_t}$$
$$L_{it}=\left(1-\alpha\right)\frac{\sigma-1}{\sigma}\left(\frac{A_i\left(\theta_t\right)}{\overline{A}\left(\theta_t,m_t\right)}\right)^{\sigma-1}\frac{P_tY_t}{W_t}.$$

Market clearing on the factor markets implies

$$K_t=m_tK_{ht}+\left(1-m_t\right)K_{lt}=\alpha\frac{\sigma-1}{\sigma}\frac{P_tY_t}{R_t}$$
$$L_t=m_tL_{ht}+\left(1-m_t\right)L_{lt}=\left(1-\alpha\right)\frac{\sigma-1}{\sigma}\frac{P_tY_t}{W_t}.$$

The equilibrium level of labor as a function of m_t can be obtained by combining the household's labor supply equation to the aggregate labor demand:

$$L_t=\left(\frac{W_t}{P_t}\right)^{\frac{1}{\nu}}=\left(\left(1-\alpha\right)\frac{\sigma-1}{\sigma}\frac{Y_t}{L_t}\right)^{\frac{1}{\nu}},$$

which delivers the equilibrium labor and output levels

$$L_t = \left[(1-\alpha)\frac{\sigma-1}{\sigma}\overline{A}(\theta_t, m_t) K_t^{\alpha}\right]^{\frac{1}{\alpha+\nu}} \tag{25}$$

$$Y_t = \left((1-\alpha)\frac{\sigma-1}{\sigma}\right)^{\frac{1-\alpha}{\alpha+\nu}} \left(\overline{A}(\theta_t, m_t) K_t^{\alpha}\right)^{\frac{1+\nu}{\alpha+\nu}}. \tag{26}$$

Finally, we may now derive expressions for individual profits:

$$\Pi_{it} = P_{it}Y_{it} - R_t K_{it} - W_t L_{it} = \frac{1}{\sigma}P_{it}Y_{it} = \frac{1}{\sigma}\left(\frac{A_i(\theta_t)}{\overline{A}(\theta_t, m_t)}\right)^{\sigma-1} P_t Y_t. \tag{27}$$

□

E.2 Multiplicity of equilibria

Proposition 2. *Consider the following condition on parameters:*

$$\frac{1+\nu}{\alpha+\nu} > \sigma - 1. \tag{14}$$

Under condition (14), there exist thresholds $B_H < B_L$ such that:
i) if $Ae^{\theta_t}K_t^{\alpha} < B_H$, the static equilibrium is unique and all firms choose low capacity, $m_t = 0$;
ii) if $Ae^{\theta_t}K_t^{\alpha} > B_L$, the static equilibrium is unique and all firms choose high capacity, $m_t = 1$;
iii) if $B_H \leqslant Ae^{\theta_t}K_t^{\alpha} \leqslant B_L$, there are three static equilibria: two in pure strategies, $m_t = 1$ and $m_t = 0$, and one in mixed strategies, $m_t \in (0,1)$.
If condition (14) is not satisfied, the static equilibrium is always unique.

Proof. Substituting in the equilibrium profit functions derived in proposition 1, the capacity decision problem becomes

$$u_{it} = \underset{u_i \in \{u_h, u_l\}}{\operatorname{argmax}} \left\{ \frac{1}{\sigma}\left(\frac{A_h(\theta_t)}{\overline{A}(\theta_t, m_t)}\right)^{\sigma-1} Y_t - f, \frac{1}{\sigma}\left(\frac{A_l(\theta_t)}{\overline{A}(\theta_t, m_t)}\right)^{\sigma-1} Y_t \right\}.$$

The capacity decision is governed by the sign of the surplus from choosing high capacity which we define as

$$\Delta\Pi(K,\theta,m) \equiv \Pi_h - \Pi_l - f = \frac{1}{\sigma}\frac{A_h(\theta)^{\sigma-1} - A_l(\theta)^{\sigma-1}}{\overline{A}(\theta,m)^{\sigma-1}}\left((1-\alpha)\frac{\sigma-1}{\sigma}\right)^{\frac{1-\alpha}{\alpha+\nu}}\left(\overline{A}(\theta,m)K^{\alpha}\right)^{\frac{1+\nu}{\alpha+\nu}} - f.$$

The economy admits a *pure strategy equilibrium with high capacity* if and only if $\Delta\Pi(K,\theta,1) \geqslant 0$, which is equivalent to

$$\frac{1}{\sigma}\frac{A_h(\theta)^{\sigma-1} - A_l(\theta)^{\sigma-1}}{A_h(\theta)^{\sigma-1}}\left((1-\alpha)\frac{\sigma-1}{\sigma}\right)^{\frac{1-\alpha}{\alpha+\nu}}\left(A_h(\theta)K^{\alpha}\right)^{\frac{1+\nu}{\alpha+\nu}} - f \geqslant 0.$$

A high equilibrium exists if and only if the following condition is satisfied:

$$Ae^{\theta}K^{\alpha} \geqslant \frac{1}{\omega}\frac{\sigma}{\sigma-1}\left(\frac{(\sigma-1)f}{1-\omega^{1-\sigma}}\right)^{\frac{\alpha+\nu}{1+\nu}} \equiv B_H.$$

Similarly, there exists a *pure strategy equilibrium with low capacity utilization* if and only if $\Delta\Pi(K,\theta,0) \leqslant 0$, which is equivalent to

$$\frac{1}{\sigma}\frac{A_h(\theta)^{\sigma-1}-A_l(\theta)^{\sigma-1}}{A_l(\theta)^{\sigma-1}}\left((1-\alpha)\frac{\sigma-1}{\sigma}\right)^{\frac{1-\alpha}{\alpha+\nu}}\left(A_l(\theta)K^{\alpha}\right)^{\frac{1+\nu}{\alpha+\nu}} - f \leqslant 0.$$

A low equilibrium exists if and only if the following condition is satisfied:

$$A(\theta)K^{\alpha} \leqslant \frac{\sigma}{\sigma-1}\left(\frac{(\sigma-1)f}{\omega^{\sigma-1}-1}\right)^{\frac{\alpha+\nu}{1+\nu}}(1-\alpha)^{-\frac{1-\alpha}{1+\nu}} \equiv B_L.$$

The thresholds are such that $B_L > B_H$ if and only $\frac{1+\nu}{\alpha+\nu} > \sigma - 1$.

Next, let us consider the mixed strategy equilibrium. Firms are indifferent between both capacities if

$$\frac{1}{\sigma}\frac{A_h(\theta)^{\sigma-1}-A_l(\theta)^{\sigma-1}}{\overline{A}(\theta,m)^{\sigma-1}}\left((1-\alpha)\frac{\sigma-1}{\sigma}\right)^{\frac{1-\alpha}{\alpha+\nu}}\left(\overline{A}(\theta,m)K^{\alpha}\right)^{\frac{1+\nu}{\alpha+\nu}} - f = 0.$$

There is a mixed strategy equilibrium if there is a solution to this equation with $m \in (0,1)$. We can rewrite this equation as

$$Ae^{\theta}K^{\alpha} = \frac{\sigma}{\sigma-1}\left(\frac{(\sigma-1)f}{\omega^{\sigma-1}-1}\right)^{\frac{\alpha+\nu}{1+\nu}}(1-\alpha)^{-\frac{1-\alpha}{1+\nu}}\left(m\left(\omega^{\sigma-1}-1\right)+1\right)^{\frac{\alpha+\nu}{1+\nu}-\frac{1}{\sigma-1}}. \tag{28}$$

If $\frac{1+\nu}{\alpha+\nu} > \sigma - 1$, the right-hand side is strictly decreasing in m and equals B_L for $m = 0$ and B_H for $m = 1$. Therefore, as long as $B_H < Ae^{\theta}K^{\alpha} < B_L$ there is a mixed strategy equilibrium in addition to the two others. Notice that the equilibrium m is decreasing in $Ae^{\theta}K^{\alpha}$.

If $\frac{1+\nu}{\alpha+\nu} \leqslant \sigma - 1$, then $B_L \leqslant B_H$ and the right-hand side of (28) is increasing in m from B_L to B_H. There is therefore a unique static equilibrium for all $Ae^{\theta}K^{\alpha}$. □

E.3 Efficiency

Proposition 3. *If $\frac{1+\nu}{\nu+\alpha} > \sigma - 1$, there exists a threshold B_{SP}, with $B_{SP} \leq B_L$, such that the planner makes all firms use the high capacity, $m_t = 1$, if $Ae^{\theta_t}K_t^{\alpha} \geqslant B_{SP}$ and firms use the low capacity, $m_t = 0$, if $Ae^{\theta_t}K_t^{\alpha} \leqslant B_{SP}$. The threshold B_{SP} is lower than B_H for σ small.*

Proof. Consider the planning problem:

$$\max_{K_{t+1},L_t,m_t} \mathbb{E}\sum_{t=0}^{\infty}\beta^t U\left(\left(m_t Y_{ht}^{\frac{\sigma-1}{\sigma}} + (1-m_t)Y_{lt}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} + (1-\delta)K_t - m_t f - K_{t+1}, L_t\right)$$

subject to

$$Y_{it} = A_i(\theta_t) K_{it}^{\alpha} L_{it}^{1-\alpha}, i \in \{h, l\}$$
$$K_t = m_t K_{ht} + (1 - m_t) K_{lt}$$
$$L_t = m_t L_{ht} + (1 - m_t) L_{lt}.$$

In the optimal allocation, the marginal products are equalized across firms:

$$\alpha \frac{Y_t^{\frac{1}{\sigma}} Y_{ht}^{1-\frac{1}{\sigma}}}{K_{ht}} = \alpha \frac{Y_t^{\frac{1}{\sigma}} Y_{lt}^{1-\frac{1}{\sigma}}}{K_{lt}} \text{ and } (1-\alpha) \frac{Y_t^{\frac{1}{\sigma}} Y_{ht}^{1-\frac{1}{\sigma}}}{L_{ht}} = (1-\alpha) \frac{Y_t^{\frac{1}{\sigma}} Y_{lt}^{1-\frac{1}{\sigma}}}{L_{lt}}.$$

Combining these two equations, we obtain the same aggregation result that we derived in proposition (1), i.e., $Y_{it} = \left(\frac{A_i(\theta_t)}{\overline{A}(\theta_t, m_t)}\right)^{\sigma} Y_t$ and $Y_t = \overline{A}(\theta_t, m_t) K_t^{\alpha} L_t^{1-\alpha}$. The planner's problem then reduces to

$$\max_{K_{t+1}, L_t, m_t} \mathbb{E} \sum_{t=0}^{\infty} \beta^t U\left(\overline{A}(\theta_t, m_t) K_t^{\alpha} L_t^{1-\alpha} + (1-\delta) K_t - m_t f - K_{t+1}, L_t\right).$$

The optimality for labor is

$$(1-\alpha) \overline{A}(\theta_t, m_t) K_t^{\alpha} L_t^{-\alpha} = L_t^{\nu}.$$

In particular, the planner's problem inherits the structure of the competitive economy, which can be solved in two stages thanks to the GHH preferences. Solving for L_t, one obtains

$$L_t = \left[(1-\alpha) \overline{A}(\theta_t, m_t) K_t^{\alpha}\right]^{\frac{1}{\alpha+\nu}},$$

a similar expression as (12) for the competitive equilibrium, except that the monopoly distortions, captured by $\frac{\sigma}{\sigma-1}$ do not distort the factor demand. Plugging this expression back in the objective function and noticing that $Y_t - \frac{L_t^{1+\nu}}{1+\nu} = \frac{\alpha+\nu}{1+\nu} Y_t$, the planner's problem can be rewritten as

$$\max_{K_{t+1}, m_t} \mathbb{E} \sum_{t=0}^{\infty} \beta^t \frac{1}{1-\gamma} \left(\frac{\alpha+\nu}{1+\nu} (1-\alpha)^{\frac{1-\alpha}{\alpha+\nu}} \left(\overline{A}(\theta_t, m_t) K_t^{\alpha}\right)^{\frac{1+\nu}{\alpha+\nu}} - m_t f + (1-\delta) K_t - K_{t+1}\right)^{1-\gamma},$$

so that the maximization over m_t boils down to maximizing production net of disutility of labor,

$$\frac{\alpha+\nu}{1+\nu} (1-\alpha)^{\frac{1-\alpha}{\alpha+\nu}} \left(\overline{A}(\theta_t, m_t) K_t^{\alpha}\right)^{\frac{1+\nu}{\alpha+\nu}} - m_t f.$$

This problem is strictly convex in m when $\frac{1+\nu}{\alpha+\nu} > \sigma - 1$, so that the planner always picks a corner solution $m_t = 0$ or $m_t = 1$. Comparing both values, the planner uses high capacity if and only if

$$\frac{\alpha+\nu}{1+\nu}(1-\alpha)^{\frac{1-\alpha}{\alpha+\nu}} \left(A_h(\theta_t) K_t^{\alpha}\right)^{\frac{1+\nu}{\alpha+\nu}} - f \geqslant \frac{\alpha+\nu}{1+\nu}(1-\alpha)^{\frac{1-\alpha}{\alpha+\nu}} \left(A_l(\theta_t) K_t^{\alpha}\right)^{\frac{1+\nu}{\alpha+\nu}},$$

which is equivalent to the condition

$$Ae^{\theta_t}K_t^{\alpha} \geqslant \left(\frac{1}{(1-\alpha)^{\frac{1-\alpha}{\alpha+\nu}}}\frac{1+\nu}{\alpha+\nu}\frac{f}{\omega^{\frac{1+\nu}{\alpha+\nu}}-1}\right)^{\frac{\alpha+\nu}{1+\nu}} \equiv B_{SP},$$

where B_{SP} is a threshold such that the planner picks the high capacity if and only if $Ae^{\theta_t}K_t^{\alpha} \geq B_{SP}$. First, let us show that $B_{SP} \leq B_L$:

$$B_{SP} \quad \leq \quad B_L \Leftrightarrow \frac{1+\nu}{\alpha+\nu}\frac{1}{\omega^{\frac{1+\nu}{\alpha+\nu}}-1} \leq \left(\frac{\sigma}{\sigma-1}\right)^{\frac{1+\nu}{\alpha+\nu}}\frac{\sigma-1}{\omega^{\sigma-1}-1}$$

which is satisfied if

$$\frac{1+\nu}{\alpha+\nu}\frac{1}{\omega^{\frac{1+\nu}{\alpha+\nu}}-1} \leq \frac{\sigma-1}{\omega^{\sigma-1}-1}.$$

The function $f(x) = \frac{1}{x}(\omega^x - 1)$ being increasing, we then conclude that $B_{SP} < B_L$ under the condition that $\frac{1+\nu}{\alpha+\nu} > \sigma - 1$.

Let us now compare B_{SP} and B_H:

$$B_{SP} \quad \leq \quad B_H \Leftrightarrow \frac{1+\nu}{\alpha+\nu}\frac{1}{\omega^{\frac{1+\nu}{\alpha+\nu}}-1} \leq \left(\frac{\sigma}{\sigma-1}\right)^{\frac{1+\nu}{\nu+\alpha}}\frac{1}{\omega^{\frac{1+\nu}{\alpha+\nu}-\sigma+1}}\frac{\sigma-1}{\omega^{\sigma-1}-1}.$$

The left-hand side is independent of σ. Since $\lim_{\sigma\to 1}\frac{\omega^{\sigma-1}-1}{\sigma-1} = \log\omega$, the right-hand side goes to ∞ as $\sigma \to 1$ from above and $B_{SP} \leq B_H$ for σ small enough. □

F Incomplete Information: Proofs - For online publication

F.1 Notation and Definitions

This section introduces some useful notation and restates various equilibrium results established in the paper when the economy is subject to an input subsidy s_{kl}, a sales subsidy s_y, a profit subsidy s_π and a lump-sum tax on the household to finance the subsidies. Under these subsidies the problem of the firm becomes

$$\Pi_{it} = \max_{Y_{it},P_{it},K_{it},L_{it}} (1+s_y)\,P_{it}Y_{it} - (1-s_{kl})\,(R_tK_{it} + W_tL_{it})$$

subject to 4 and 5 and where the capacity choice is such that

$$u_j = u_h \Longleftrightarrow \mathbb{E}_\theta\left[U_c\left(C_t, L_t\right)\left(\left(1+s_\pi\right)\left(\Pi_h\left(K_t,\theta_t,m_t\right) - \Pi_l\left(K_t,\theta_t,m_t\right)\right) - f\right) \mid \theta_{t-1}, v_{jt}\right] \geqslant 0.$$

Notation

As introduced in the main text, we denote by $\overline{A}$ the endogenous aggregate TFP of the economy, with

$$\overline{A}(\theta, m) \equiv Ae^{\theta}\Omega(m),$$

where $\Omega(m) \equiv \left(m\left(\omega^{\sigma-1}-1\right)+1\right)^{\frac{1}{\sigma-1}}$ is an average of capacity across firms. The equilibrium level of output and labor as a function of K, θ and m is

$$Y(K,\theta,m) \equiv \left[(1-\alpha)\frac{\sigma-1}{\sigma}\frac{1+s_y}{1-s_{kl}}\right]^{\frac{1-\alpha}{\alpha+\nu}} \overline{A}(\theta,m)^{\frac{1+\nu}{\alpha+\nu}} K^{\alpha\frac{1+\nu}{\alpha+\nu}},$$

and labor

$$L(K,\theta,m) \equiv \left[(1-\alpha)\frac{\sigma-1}{\sigma}\frac{1+s_y}{1-s_{kl}}\right]^{\frac{1}{\alpha+\nu}} \overline{A}(\theta,m)^{\frac{1}{\alpha+\nu}} K^{\frac{\alpha}{\alpha+\nu}}.$$

The corresponding rental rate of capital is

$$R(K,\theta,m) \equiv \alpha\frac{\sigma-1}{\sigma}\frac{1+s_y}{1-s_{kl}}\frac{Y(K,\theta,m)}{K}.$$

To lighten notation, it is also useful to introduce the gross output level net of fixed costs and depreciation

$$y(K,\theta,m) \equiv Y(K,\theta,m) + (1-\delta)K - mf,$$

and the corresponding net interest rate

$$r(K,\theta,m) \equiv R(K,\theta,m) + 1 - \delta.$$

Finally, we denote the equilibrium output level for firms of type h and l by

$$Y_h(K,\theta,m) \equiv \frac{\omega^{\sigma}}{\Omega(m)^{\sigma}}Y(K,\theta,m) \text{ and } Y_l(K,\theta,m) \equiv \frac{1}{\Omega(m)^{\sigma}}Y(K,\theta,m),$$

and profit rates

$$\Pi_h(K,\theta,m) \equiv \frac{1}{\sigma}\frac{1+s_y}{1-s_{kl}}\frac{\omega^{\sigma-1}}{\Omega(m)^{\sigma-1}}Y(K,\theta,m) \text{ and } \Pi_l(K,\theta,m) \equiv \frac{1}{\sigma}\frac{1+s_y}{1-s_{kl}}\frac{1}{\Omega(m)^{\sigma-1}}Y(K,\theta,m).$$

We sometimes abuse notation in part F.4 of the proofs, once conditions for existence and uniqueness of a solution to the global game have been established, by writing $Y(K,\theta_{-1},\theta) = Y(K,\theta,m(K,\theta_{-1},\theta))$, $R(K,\theta_{-1},\theta) = R(K,\theta,m(K,\theta_{-1},\theta))$, and so on. Furthermore, we use the vector notation $\boldsymbol{\theta} = (\theta_{-1},\theta)'$ in several parts of the proofs to avoid spelling out the entire state space.

Assumptions and Definitions

Our existence and uniqueness proofs require the value and policy functions to be bounded. We thus restrict the fundamental to remain between two bounds $\left[\underline{\theta}, \overline{\theta}\right]$, chosen large enough that they contain most of the ergodic distribution of θ.[34]

Definition 3. Let $\Theta = \left[\underline{\theta}, \overline{\theta}\right]$. The fundamental θ follows the autoregressive process

$$\theta = \min\left(\max\left(\rho\theta_{-1} + e_t^{\theta}, \underline{\theta}\right), \overline{\theta}\right),$$

and we denote its transition density $\pi\left(\theta, d\theta'\right) = \Pr\left\{\theta_{t+1} \in \left[\theta', \theta' + d\theta'\right] \mid \theta_t = \theta\right\}$.

Definition 4. Let $\mathbb{K} = \left[0, \overline{K}\right]$ where $\overline{K}$ is implicitly defined by

$$Y\left(\overline{K}, \overline{\theta}, 1\right) + (1-\delta)\,\overline{K} - \frac{L\left(\overline{K}, \overline{\theta}, 1\right)^{1+\nu}}{1+\nu} = \overline{K}.$$

Definition 4 defines the set in which the stock of capital lies and $\overline{K}$ which corresponds to the maximal output ever achievable is an upper bound on capital. The upper bound $\overline{K}$ exists and is unique under assumption 1 below.

Assumption 1. *The parameters satisfy*

$$\frac{\sigma-1}{\sigma}\frac{1-\alpha}{1+\nu}\frac{1+s_y}{1-s_{kl}} \leqslant 1.$$

Assumption 1 is a feasibility condition required by the GHH preferences. It guarantees that total production, $y\left(K, \boldsymbol{\theta}\right)$, net of minimum consumption, $(1+\nu)^{-1}L\left(K, \boldsymbol{\theta}\right)^{1+\nu}$, is positive, so that there exists a solution to the static equilibrium in production.

Assumption 2. *The lower bound $\underline{\theta}$ is chosen sufficiently small that there exists $K^- > 0$ such that $y\left(K^-, \underline{\theta}, \underline{\theta}\right) - \frac{L\left(K^-, \underline{\theta}, \underline{\theta}\right)^{1+\nu}}{1+\nu} > K^-$ and $\beta\mathbb{E}\left[r\left(K^-, \underline{\theta}, \theta'\right) \mid \theta = \underline{\theta}\right] \leqslant 1$.*

Assumption 2 is from Coleman (1991) and is necessary to show the existence of a non-zero equilibrium. Note that there always exists a K^- that satisfies the first part of the definition given our choice for the production function and $m \simeq 0$. The key requirement comes from the second part and can be achieved by assuming that $\underline{\theta}$ is sufficiently low.

[34]For arbitrarily large bounds, this restriction has no bearing on our quantitative results and the Bayesian updating rules for untruncated normals, that we use to update private beliefs in the static global game, provide an arbitrarily good approximation to the true beliefs with truncated normals for the relevant part of the ergodic distribution of θ.

F.2 Main proposition

Proposition 4 (Full). *Under Assumption 1-2, for γ_v large and ω sufficiently close to 1 such that i) approximation (31) holds, ii) parameters satisfy*

$$\frac{\sqrt{\gamma_v}}{\gamma_\theta} > \frac{1}{\sqrt{2\pi}} \frac{\omega^{\sigma-1} - 1}{\sigma - 1}, \tag{18}$$

and

$$\frac{1 - \alpha^{\frac{1+\nu}{\alpha+\nu}}}{\alpha^{\frac{1+\nu}{\alpha+\nu}}} \geqslant \frac{1}{\sqrt{2\pi}} \frac{\omega^{\sigma-1} - 1}{\sigma - 1} \frac{\gamma_\theta + \gamma_v}{\sqrt{\gamma_v} - \frac{1}{\sqrt{2\pi}} \frac{\omega^{\sigma-1}-1}{\sigma-1} \gamma_\theta} \tag{29}$$

and iii) $y(K, \theta_{-1}, \theta) - \frac{L(K,\theta_{-1},\theta)^{1+\nu}}{1+\nu}$ is weakly increasing in K, there exists a unique dynamic equilibrium. The equilibrium capacity decision takes the form of a continuous cutoff $\hat{v}(K, \theta_{-1})$ such that firm j invests if and only if $v_j \geqslant \hat{v}(K, \theta_{-1})$. Furthermore, the cutoff is a decreasing function of its arguments.

Proof. We prove this proposition in several steps. In a first step, we show in section F.3 below that, for a small departure from complete information, i.e., when γ_v is large, risk becomes irrelevant for firms and the stochastic discount factor drops out of their capacity decision (lemma A1). In that case, we can solve the global game independently from the rest of the dynamic equilibrium. In proposition A1, we show that there exists a unique equilibrium to the global game under condition (18) and that the equilibrium capacity decision takes the form of a cutoff strategy $\hat{v}(K, \theta_{-1})$ such that firms choose high capacity if and only if they receive a signal v_j above that threshold.

Using the resulting $\hat{v}$ from the global game, we show the existence and uniqueness of the dynamic equilibrium in section F.4. Proposition A2 establishes existence under several additional assumptions. First, assumption 1 ensures that the firm's decision is well defined and bounded by putting an upper bound on the subsidies that they receive. It is trivially satisfied for a competitive economy without government subsidies, but the proposition shows how the result extends to economies with input, sales and profit subsidies. Assumption 2 is relatively mild as it only requires us to choose a sufficiently low bound $\underline{\theta}$. Second, the condition that ω is close to 1 ensures that the Euler equation is a monotone operator, which our proof uses to prove existence. In particular, monotonicity requires that R is a nonincreasing function of K, which (29) guarantees (lemma (A2).(v)), and that output net of fixed costs minus labor, $y(K, \theta_{-1}, \theta) - \frac{L(K,\theta_{-1},\theta)^{1+\nu}}{1+\nu}$, is nondecreasing in K, a property satisfied for ω sufficiently close to 1 (lemma (A2).(iv)). These last two conditions are sufficient but not necessary and could be relaxed in practice.[35] Notice also that the proposition provides the existence of a strictly positive equilibrium, in the sense that consumption is non-zero whenever $K > 0$.

Proposition A3 finally establishes uniqueness of the (strictly positive) equilibrium under the same conditions on the parameters. ☐

[35] For instance, one could check numerically that functions R and y have the correct monotonicity properties for the proof to go through.

F.3 Global game

Description

This section describes the solution of the static game played every period between the intermediate goods producers.

The decision of intermediate producer j to operate at high capacity over low capacity is determined by the sign of the surplus,

$$\Delta\Pi\left(K,\theta_{-1},v_j,m\right)=\mathbb{E}_\theta\left[U_C\left(C,L\right)\left(\left(1+s_\pi\right)\left(\Pi_h\left(K,\theta,m\right)-\Pi_l\left(K,\theta,m\right)\right)-f\right)\mid\theta_{-1},v_j\right],$$

such that producer j chooses high capacity if and only if $\Delta\Pi\geqslant 0$. In equilibrium, C and L are functions of the aggregate state space (K,θ_{-1},θ), which we sometimes write $(K,\boldsymbol{\theta})$ with vector $\boldsymbol{\theta}=(\theta_{-1},\theta)'$. Anticipating on the rest of the proof, let us denote the inverse marginal utility of consumption $P\left(K,\boldsymbol{\theta}\right)=\left[U_C\left(C\left(K,\boldsymbol{\theta}\right),L\left(K,\boldsymbol{\theta}\right)\right)\right]^{-1}$. Substituting with the equilibrium value of profits, the expected surplus from operating at high vs. low capacity for firm j with a perceived mass of entrants $m\left(K,\theta,\theta_{-1}\right)$ is

$$\Delta\Pi\left(K,\theta_{-1},v_j,m\right)=\mathbb{E}_\theta\left[\frac{1}{P\left(K,\boldsymbol{\theta}\right)}\left(\frac{1}{\sigma}\frac{\left(1+s_\pi\right)\left(1+s_y\right)}{1-s_{kl}}\frac{\omega^{\sigma-1}-1}{\Omega\left(m\right)^{\sigma-1}}Y\left(K,\theta,m\right)-f\right)\mid\theta_{-1},v_j\right]. \tag{30}$$

The presence of the stochastic discount factor in the problem of the firm introduces an additional complication in comparison to standard global games without general equilibrium effects. Fortunately, under the assumption that γ_v is large, i.e., for a small deviation from common knowledge, the stochastic discount factor drops out of the equation and is, thus, asymptotically irrelevant. The outcome of the global game may thus be approximated by a simpler problem, which we describe below.

Approximation

Lemma A1. *Let $P:(K,\boldsymbol{\theta})\in\mathbb{K}\times\Theta^2\to\mathbb{R}$ bounded, continuous, positive and bounded away from 0 over $\left[\underline{K},\overline{K}\right]\times\Theta^2$ for all $\underline{K}>0$. Then for all $\hat{v}\in\mathbb{R}$,*

$$\left|\mathbb{E}_\theta\left[\frac{\left(1+s_\pi\right)\left(\Pi_h\left(K,\theta,m\right)-\Pi_l\left(K,\theta,m\right)\right)-f}{P\left(K,\theta_{-1},\theta\right)}\mid\theta_{-1},v_j\right]-\mathbb{E}_\theta\left[\frac{\left(1+s_\pi\right)\left(\Pi_h\left(K,\theta,m\right)-\Pi_l\left(K,\theta,m\right)\right)-f}{P\left(K,\theta_{-1},\frac{\gamma_\theta\rho\theta_{-1}+\gamma_v v_j}{\gamma_\theta+\gamma_v}\right)}\mid\theta_{-1},v_j\right]\right|\underset{\gamma_v\to\infty}{\longrightarrow}0.$$

Furthermore, the convergence is uniform over $\left[\underline{K},\overline{K}\right]\times\Theta^2$.

Proof. To lighten notation, denote

$$\Delta Y\left(K,\theta,m\right)\equiv\left(1+s_\pi\right)\left(\Pi_h\left(K,\theta,m\right)-\Pi_l\left(K,\theta,m\right)\right)-f=\frac{1}{\sigma}\frac{\left(1+s_\pi\right)\left(1+s_y\right)}{1-s_{kl}}\frac{\omega^{\sigma-1}-1}{\Omega\left(m\right)^{\sigma-1}}Y\left(K,\theta,m\right)-f.$$

Since $\theta \mid \theta_{-1}, v_j \sim \mathcal{N}\left(\frac{\gamma_\theta \rho \theta_{-1}+\gamma_v v_j}{\gamma_\theta+\gamma_v}, \frac{1}{\gamma_\theta+\gamma_v}\right)$, we can control the above expression as follows:

$$\left|\mathbb{E}_\theta\left[\frac{\Delta Y\left(K,\theta,m\right)}{P\left(K,\theta_{-1},\theta\right)} \mid \theta_{-1}, v_j\right] - \mathbb{E}_\theta\left[\frac{\Delta Y\left(K,\theta,m\right)}{P\left(K,\theta_{-1},\frac{\gamma_\theta \rho \theta_{-1}+\gamma_v v_j}{\gamma_\theta+\gamma_v}\right)} \mid \theta_{-1}, v_j\right]\right|$$
$$\leqslant \overline{\Delta Y}\frac{\overline{P}_\theta}{\inf\limits_\theta P\left(K,\theta_{-1},\theta\right)^2}\mathbb{E}_\theta\left[\left|\theta - \frac{\gamma_\theta \rho \theta_{-1}+\gamma_v v_j}{\gamma_\theta+\gamma_v}\right| \mid \theta_{-1}, v_j\right] \leqslant \overline{\Delta Y}\frac{\overline{P}_\theta}{\inf\limits_\theta P\left(K,\theta_{-1},\theta\right)^2}\frac{1}{\gamma_\theta+\gamma_v}\sqrt{\frac{2}{\pi}},$$

where $\overline{P}_\theta$ and $\overline{\Delta Y}$ are the modulus of uniform continuity of P and ΔY along θ. Therefore, we have pointwise convergence and uniform convergence on all segments $\left[\underline{K},\overline{K}\right]$ with $\underline{K} > 0$ with the uniform bound $\overline{\Delta Y}\frac{\overline{P}_\theta}{\inf\limits_{[\underline{K},\overline{K}]\times\Theta^2} P(\underline{K},\theta_{-1},\theta)^2}\frac{1}{\gamma_\theta+\gamma_v}\sqrt{\frac{2}{\pi}}$. □

Choosing the bound $\underline{K}$ low enough that $\left[\underline{K},\overline{K}\right]$ contains all the stocks of capital ever visited along the equilibrium path, the above lemma tells us that, in the limit as $\gamma_v \to \infty$, we can approximate the surplus from choosing high capacity by the simpler expression:

$$\Delta\tilde{\Pi}\left(K,\theta_{-1},v_j,m\right) \equiv \frac{1}{P\left(K,\theta_{-1},\frac{\gamma_\theta \rho \theta_{-1}+\gamma_v v_j}{\gamma_\theta+\gamma_v}\right)}\mathbb{E}_\theta\left[\Delta Y\left(K,\theta,m\right) \mid \theta_{-1}, v_j\right].$$

The intuition behind this expression is that, as $\gamma_v \to \infty$, consumption risk vanishes and becomes irrelevant to firms. This does not mean, however, that uncertainty is unimportant: the firms' decision is then entirely driven by strategic concerns, captured by $E_\theta\left[\Delta Y \mid \theta_{-1}, v_j\right]$, in which uncertainty plays a crucial role.

We focus, from now on, on the cases where γ_v is high and the above approximation holds. Under that assumption, the decision of firm j is

$$u_j = u_h \Leftrightarrow \mathbb{E}_\theta\left[\Delta Y\left(K,\theta,m\right) \mid \theta_{-1}, v_j\right] \geqslant 0. \tag{31}$$

It is worth noting here that our numerical results suggest that the approximation is very accurate: under our benchmark calibration, the solutions of the global game using expression (30) and (31) are virtually indistinguishable.

Existence and Uniqueness

Proposition A1. *For γ_v large enough that approximation (31) holds and*

$$\frac{\sqrt{\gamma_v}}{\gamma_\theta} > \frac{1}{\sqrt{2\pi}}\frac{\omega^{\sigma-1}-1}{\sigma-1}, \tag{18}$$

then the optimal capacity decision takes the form of a unique cutoff strategy $\hat{v}\left(K,\theta_{-1}\right)$ such that firm j chooses high capacity if and only if $v_j \geqslant \hat{v}$.

Proof. Fix $K \in \mathbb{K}$ and $\theta_{-1} \in \Theta$. Under the hypothesis that γ_v is large enough that the approximation (31) holds, firm j chooses the high capacity if and only if

$$\Delta\tilde{\Pi}(K, \theta_{-1}, v_j, m) \geqslant 0 \Leftrightarrow \mathbb{E}_\theta \left[\frac{1}{\sigma} \frac{(1+s_\pi)(1+s_y)}{1-s_{kl}} \frac{\omega^{\sigma-1}-1}{\Omega(m)^{\sigma-1}} Y(K, \theta, m) - f \mid \theta_{-1}, v_j\right] \geqslant 0.$$

The proof proceeds in two steps. In a first step, we start solving the game by iterated deletion of dominated strategies. In a second step, we provide conditions under which this procedure converges towards a unique equilibrium.

■ Case $\frac{1+\nu}{\alpha+\nu} > \sigma - 1$

Step 1. To lighten notation, denote

$$\begin{aligned}
\Delta Y(K, \theta, m) &\equiv \frac{1}{\sigma} \frac{(1+s_\pi)(1+s_y)}{1-s_{kl}} \frac{\omega^{\sigma-1}-1}{\Omega(m)^{\sigma-1}} Y(K, \theta, m) - f \\
&= \frac{1}{\sigma} \frac{(1+s_\pi)(1+s_y)}{1-s_{kl}} \left(\omega^{\sigma-1}-1\right) \left[(1-\alpha) \frac{\sigma-1}{\sigma} \frac{1+s_y}{1-s_{kl}}\right]^{\frac{1-\alpha}{\alpha+\nu}} \left(Ae^\theta K^\alpha\right)^{\frac{1+\nu}{\alpha+\nu}} \Omega(m)^{\frac{1+\nu}{\alpha+\nu}-\sigma+1} - f.
\end{aligned}$$

When $\frac{1+\nu}{\alpha+\nu} > \sigma - 1$, ΔY is increasing in all its arguments. We proceed by iterated deletion of dominated strategies. Initialize the recursion by defining $\hat{v}^0 = \infty$ and $\hat{v}_0 = -\infty$, such that it is dominant to choose the high capacity for $v_j \geqslant \hat{v}^0$ and dominant to choose low capacity for $v_j \leqslant \hat{v}_0$. We now define $\hat{v}^1(K, \theta_{-1})$ such that

$$\mathbb{E}_\theta\left[\Delta Y(K, \theta, 0) \mid \theta_{-1}, \hat{v}^1\right] = \mathbb{E}_\theta\left[\Delta Y\left(K, \theta, \Phi\left(\sqrt{\gamma_v}\left(\theta - \hat{v}^0\right)\right)\right) \mid \theta_{-1}, \hat{v}^1\right] = 0,$$

which means that it is dominant to choose high capacity for all firms j such that $v_j \geqslant \hat{v}^1$, even if no one else did. Symmetrically, we can define $\hat{v}_1(K, \theta_{-1})$ such that

$$\mathbb{E}_\theta\left[\Delta Y(K, \theta, 1) \mid \theta_{-1}, \hat{v}_1\right] = \mathbb{E}_\theta\left[\Delta Y\left(K, \theta, \Phi\left(\sqrt{\gamma_v}\left(\theta - \hat{v}_0\right)\right)\right) \mid \theta_{-1}, \hat{v}_1\right] = 0,$$

such that it is dominant to choose low capacity for all firms j such that $v_j \leqslant \hat{v}_1$ even if all other firms choose the high capacity. By the properties of ΔY, we must have $\hat{v}_0 < \hat{v}_1 \leqslant \hat{v}^1 < \hat{v}^0$. This establishes the first iteration of our procedure. By induction, let $n \geqslant 2$ and assume that $\hat{v}_0 < \ldots < \hat{v}_{n-1} \leqslant \hat{v}^{n-1} < \ldots < \hat{v}^0$ such that it is dominant to choose high capacity if $v_j \geqslant \hat{v}^{n-1}$ and dominant to choose low for $v_j \leqslant \hat{v}_{n-1}$. Define $\hat{v}^n$ and $\hat{v}_n$ such that

$$\begin{aligned}
\mathbb{E}_\theta\left[\Delta Y\left(K, \theta, \Phi\left(\sqrt{\gamma_v}\left(\theta - \hat{v}^{n-1}\right)\right)\right) \mid \theta_{-1}, \hat{v}^n\right] = 0, \\
\mathbb{E}_\theta\left[\Delta Y\left(K, \theta, \Phi\left(\sqrt{\gamma_v}\left(\theta - \hat{v}_{n-1}\right)\right)\right) \mid \theta_{-1}, \hat{v}_n\right] = 0.
\end{aligned}$$

By induction, $\hat{v}^{n-1} \geqslant \hat{v}_{n-1}$, so that $\Phi\left(\sqrt{\gamma_v}\left(\theta - \hat{v}^{n-1}\right)\right) \leqslant \Phi\left(\sqrt{\gamma_v}\left(\theta - \hat{v}_{n-1}\right)\right)$ and $\hat{v}^n \geqslant \hat{v}_n$. Also, since $\hat{v}^{n-1} < \hat{v}^{n-2}$, then $\Phi\left(\sqrt{\gamma_v}\left(\theta - \hat{v}^{n-1}\right)\right) > \Phi\left(\sqrt{\gamma_v}\left(\theta - \hat{v}^{n-2}\right)\right)$ and $\hat{v}^n < \hat{v}^{n-1}$. Symmetrically, we have $\hat{v}_n > \hat{v}_{n-1}$. This establishes the recursion.

Sequence $(\hat{v}_n)_{n\geqslant 0}$ is a strictly increasing bounded sequence, therefore it converges. Denote $\hat{v}_\infty$ its limit: $\hat{v}_n \underset{n\to\infty}{\rightarrow} \hat{v}_\infty$. Symmetrically, establish that $(\hat{v}^n)_{n\geqslant 0}$ converges towards some limit $\hat{v}^\infty \geqslant \hat{v}_\infty$. By continuity of ΔY, we have

$$\mathbb{E}_\theta\left[\Delta Y\left(K,\theta,\Phi\left(\sqrt{\gamma_v}\left(\theta-\hat{v}^\infty\right)\right)\right) \mid \theta_{-1},\hat{v}^\infty\right]=0 \text{ and } \mathbb{E}_\theta\left[\Delta Y\left(K,\theta,\Phi\left(\sqrt{\gamma_v}\left(\theta-\hat{v}_\infty\right)\right)\right) \mid \theta_{-1},\hat{v}_\infty\right]=0.$$

Step 2. Define $H\left(K,\theta_{-1},\hat{v}\right) \equiv \mathbb{E}_\theta\left[\Delta Y\left(K,\theta,\Phi\left(\sqrt{\gamma_v}\left(\theta-\hat{v}\right)\right)\right) \mid \theta_{-1},\hat{v}\right]$. We now provide sufficient conditions such that the implicit equation in $\hat{v}$,

$$H\left(K,\theta_{-1},\hat{v}\right)=0,$$

has a unique solution $\hat{v}\left(K,\theta_{-1}\right)$. In particular, this condition is satisfied if H is strictly increasing in $\hat{v}$. Since $\theta \mid \theta_{-1}, v_j=\hat{v} \sim \mathcal{N}\left(\frac{\gamma_\theta\rho\theta_{-1}+\gamma_v\hat{v}}{\gamma_\theta+\gamma_v},\frac{1}{\gamma_\theta+\gamma_v}\right)$,

$$\begin{aligned}
H\left(K,\theta_{-1},\hat{v}\right) &= \mathbb{E}_\theta\left[\Delta Y\left(K,\theta,\Phi\left(\sqrt{\gamma_v}\left(\theta-\hat{v}\right)\right)\right) \mid \theta_{-1},\hat{v}\right]\\
&= \mathbb{E}_\varepsilon\left[c_0\left(Ae^{\frac{\gamma_\theta\rho\theta_{-1}+\gamma_v\hat{v}}{\gamma_\theta+\gamma_v}+\varepsilon}K^\alpha\right)^{\frac{1+\nu}{\alpha+\nu}}\Omega\left(\Phi\left(\sqrt{\gamma_v}\left(\frac{\gamma_\theta\rho\theta_{-1}+\gamma_v\hat{v}}{\gamma_\theta+\gamma_v}+\varepsilon-\hat{v}\right)\right)\right)^{\frac{1+\nu}{\alpha+\nu}-\sigma+1}-f\right]
\end{aligned}$$

where $c_0=\frac{1}{\sigma}\frac{(1+s_\pi)(1+s_y)}{1-s_{kl}}\left(\omega^{\sigma-1}-1\right)\left[(1-\alpha)\frac{\sigma-1}{\sigma}\frac{1+s_y}{1-s_{kl}}\right]^{\frac{1-\alpha}{\alpha+\nu}}$ and $\varepsilon=\theta-\frac{\gamma_\theta\rho\theta_{-1}+\gamma_v\hat{v}}{\gamma_\theta+\gamma_v}\sim\mathcal{N}\left(0,\frac{1}{\gamma_\theta+\gamma_v}\right)$. Compute the derivative:

$$\frac{\partial \log H}{\partial \hat{v}}=\frac{1+\nu}{\alpha+\nu}\frac{\gamma_v}{\gamma_\theta+\gamma_v}+\frac{\partial}{\partial \hat{v}}\log H_2\left(\theta_{-1},\hat{v}\right)$$

where $H_2\left(\theta_{-1},\hat{v}\right)=\mathbb{E}_\varepsilon\left[e^{\frac{1+\nu}{\alpha+\nu}\varepsilon}\Omega\left(\Phi\left(\sqrt{\gamma_v}\left(\frac{\gamma_\theta(\rho\theta_{-1}-\hat{v})}{\gamma_\theta+\gamma_v}+\varepsilon\right)\right)\right)^{\frac{1+\nu}{\alpha+\nu}-\sigma+1}\right]$. Compute the last term,

$$\begin{aligned}
\frac{\partial H_2}{\partial \hat{v}} = -\frac{\sqrt{\gamma_v}\gamma_\theta}{\gamma_\theta+\gamma_v}\left(\frac{1+\nu}{\alpha+\nu}-\sigma+1\right)\mathbb{E}_\varepsilon\Bigg[&\phi\left(\sqrt{\gamma_v}\left(\frac{\gamma_\theta\left(\rho\theta_{-1}-\hat{v}\right)}{\gamma_\theta+\gamma_v}+\varepsilon\right)\right)\frac{\Omega'\left(\Phi\left(\sqrt{\gamma_v}\left(\frac{\gamma_\theta(\rho\theta_{-1}-\hat{v})}{\gamma_\theta+\gamma_v}+\varepsilon\right)\right)\right)}{\Omega\left(\Phi\left(\sqrt{\gamma_v}\left(\frac{\gamma_\theta(\rho\theta_{-1}-\hat{v})}{\gamma_\theta+\gamma_v}+\varepsilon\right)\right)\right)}\\
&\times e^{\frac{1+\nu}{\alpha+\nu}\varepsilon}\Omega\left(\Phi\left(\sqrt{\gamma_v}\left(\frac{\gamma_\theta\left(\rho\theta_{-1}-\hat{v}\right)}{\gamma_\theta+\gamma_v}+\varepsilon\right)\right)\right)^{\frac{1+\nu}{\alpha+\nu}-\sigma+1}\Bigg]
\end{aligned}$$

so that $\left|\frac{\partial H_2}{\partial \hat{v}}\right| \leqslant \frac{\sqrt{\gamma_v}\gamma_\theta}{\gamma_\theta+\gamma_v}\left(\frac{1+\nu}{\alpha+\nu}-\sigma+1\right)\frac{1}{\sqrt{2\pi}}\overline{\frac{\Omega'}{\Omega}}H_2\left(\theta_{-1},\hat{v}\right)$. Since $\left|\frac{\Omega'(m)}{\Omega(m)}\right| \leqslant \frac{\omega^{\sigma-1}-1}{\sigma-1}$, we have

$$\left|\frac{\partial \log H_2}{\partial \hat{v}}\right| \leqslant \frac{\sqrt{\gamma_v}\gamma_\theta}{\gamma_\theta+\gamma_v}\left(\frac{1+\nu}{\alpha+\nu}-\sigma+1\right)\frac{1}{\sqrt{2\pi}}\frac{\omega^{\sigma-1}-1}{\sigma-1}.$$

We may now conclude that

$$\frac{\partial \log H}{\partial \hat{v}} \geqslant \frac{1+\nu}{\alpha+\nu}\frac{\gamma_v}{\gamma_\theta+\gamma_v}-\frac{\sqrt{\gamma_v}\gamma_\theta}{\gamma_\theta+\gamma_v}\frac{1}{\sqrt{2\pi}}\left(\frac{1+\nu}{\alpha+\nu}-\sigma+1\right)\frac{\omega^{\sigma-1}-1}{\sigma-1}.$$

Therefore, a sufficient condition that guarantees that H is strictly increasing in $\hat{v}$ is

$$\frac{\sqrt{\gamma_v}}{\gamma_\theta} > \frac{1}{\sqrt{2\pi}} \frac{\omega^{\sigma-1}-1}{\sigma-1}.$$

Note in addition that $H \underset{\hat{v}\to\infty}{\longrightarrow} \infty$ and $H \underset{\hat{v}\to-\infty}{\longrightarrow} -\infty$, therefore there exists a unique solution $\hat{v}(K,\theta_{-1})$ to the equation $H(K,\theta_{-1},\hat{v}(K,\theta_{-1}))=0$.

Conclusion. Under the condition (18), there exists a unique solution to the equation

$$\mathbb{E}_\theta\left[\Delta Y\left(K,\theta,\Phi\left(\sqrt{\gamma_v}\left(\theta-\hat{v}\right)\right)\right) \mid \theta_{-1},\hat{v}\right]=0,$$

which is satisfied by both $\hat{v}^\infty$ and $\hat{v}_\infty$. Therefore, $\hat{v}^\infty=\hat{v}_\infty=\hat{v}(K,\theta_{-1})$ and the solution to the global game is the unique cutoff strategy $\hat{v}(K,\theta_{-1})$ such that firm j chooses high capacity if and only if $v_j \geqslant \hat{v}(K,\theta_{-1})$.

■ Case $\frac{1+\nu}{\alpha+\nu} \leqslant \sigma-1$

In the case that the condition for multiplicity is not satisfied, the proof is similar but easier since there is strategic substitutability between firms. By iterated deletion of dominant strategies, define the monotone sequences $(\hat{v}_n)_{n\geqslant 0}$ and $(\hat{v}^n)_{n\geqslant 0}$ by

$$\mathbb{E}_\theta\left[\Delta Y\left(K,\theta,\Phi\left(\sqrt{\gamma_v}\left(\theta-\hat{v}_{n-1}\right)\right)\right) \mid \theta_{-1},\hat{v}^n\right]=0,$$
$$\mathbb{E}_\theta\left[\Delta Y\left(K,\theta,\Phi\left(\sqrt{\gamma_v}\left(\theta-\hat{v}^{n-1}\right)\right)\right) \mid \theta_{-1},\hat{v}_n\right]=0.$$

Then, function $\mathbb{E}_\theta\left[\Delta Y\left(K,\theta,\Phi\left(\sqrt{\gamma_v}\left(\theta-\hat{v}\right)\right)\right) \mid \theta_{-1},\hat{v}\right]$ is strictly increasing in $\hat{v}$ without additional restrictions on the parameter. Conclude as in the previous case. □

Regularity

In this section, we establish a number of regularity conditions and properties of $\hat{v}$, m, $\overline{A}$ and Y.

Lemma A2. *Under the conditions of proposition A1, (i) $\hat{v}(K,\theta_{-1})$ is continuous and weakly decreasing in K and θ_{-1} and such that*

$$-\frac{\alpha\frac{1}{K}}{\frac{\sqrt{\gamma_v}}{\gamma_\theta+\gamma_v}\left[\sqrt{\gamma_v}-\frac{1}{\sqrt{2\pi}}\frac{\omega^{\sigma-1}-1}{\sigma-1}\gamma_\theta\right]} \leqslant \frac{\partial\hat{v}}{\partial K}(K,\theta_{-1}) \leqslant 0,$$

and

$$-\frac{\rho\gamma_\theta}{\sqrt{\gamma_v}\left(\sqrt{\gamma_v}-\frac{1}{\sqrt{2\pi}}\frac{\omega^{\sigma-1}-1}{\sigma-1}\gamma_\theta\right)} \leqslant \frac{\partial\hat{v}}{\partial\theta_{-1}}(K,\theta_{-1}) \leqslant 0.$$

(ii) $m(K,\theta_{-1},\theta)$ and $\overline{A}(K,\theta_{-1},\theta)$ are bounded, continuous and weakly increasing in all their arguments, (iii) $y(K,\theta_{-1},\theta)$ is bounded, continuous and, for ω sufficiently close to 1, increasing in K, (iv) if in addition assumption 1 is verified, $y(K,\theta_{-1},\theta)-\frac{L(K,\theta_{-1},\theta)^{1+\nu}}{1+\nu}$ is increasing in K, (v)

if parameters are such that

$$\frac{1-\alpha\frac{1+\nu}{\alpha+\nu}}{\alpha\frac{1+\nu}{\alpha+\nu}} \geqslant \frac{1}{\sqrt{2\pi}}\frac{\omega^{\sigma-1}-1}{\sigma-1}\frac{\gamma_\theta+\gamma_v}{\sqrt{\gamma_v}-\frac{1}{\sqrt{2\pi}}\frac{\omega^{\sigma-1}-1}{\sigma-1}\gamma_\theta},$$

then $R(K,\theta_{-1},\theta)$ *is nonincreasing in* K*.*

Proof. (i) *Continuity and monotonicity of* $\hat{v}(K,\theta_{-1})$. Cutoff $\hat{v}(K,\theta_{-1})$ is implicitly defined by the function $\Delta\tilde{\Pi}\left(K,\theta_{-1},\hat{v},\Phi\left(\sqrt{\gamma_v}\left(\theta-\hat{v}\right)\right)\right)=0$ which is a continuously differentiable function of K, θ_{-1} and $\hat{v}$. Under the conditions of proposition A1, $\frac{d}{d\hat{v}}\Delta\tilde{\Pi}>0$, so the implicit function theorem tells us that $\hat{v}$ is continuous and differentiable in a neighborhood of (K,θ_{-1}). In addition, the implicit function theorem tells us that

$$\frac{\partial\hat{v}}{\partial K}=-\left(\frac{\partial H}{\partial\hat{v}}\right)^{-1}\frac{\partial H}{\partial K}\text{ and }\frac{\partial\hat{v}}{\partial\theta_{-1}}=-\left(\frac{\partial H}{\partial\hat{v}}\right)^{-1}\frac{\partial H}{\partial\theta_{-1}},$$

where

$$\begin{aligned}
&H\left(K,\theta_{-1},\hat{v}\right)\equiv\Delta\tilde{\Pi}\left(K,\theta_{-1},\hat{v},\Phi\left(\sqrt{\gamma_v}\left(\theta-\hat{v}\right)\right)\right)\\
&=-f+H_0\int\left(Ae^{\frac{\gamma_\theta\rho\theta_{-1}+\gamma_v\hat{v}}{\gamma_\theta+\gamma_v}+\varepsilon}K^\alpha\right)^{\frac{1+\nu}{\alpha+\nu}}\times\\
&\Omega\left(\Phi\left(\sqrt{\gamma_v}\left(\frac{\gamma_\theta}{\gamma_\theta+\gamma_v}\left(\rho\theta_{-1}-\hat{v}\right)+\varepsilon\right)\right)\right)^{\frac{1+\nu}{\alpha+\nu}-\sigma+1}\sqrt{\gamma_\theta+\gamma_v}\phi\left(\sqrt{\gamma_\theta+\gamma_v}\varepsilon\right)d\varepsilon,
\end{aligned}$$

and $H_0=\frac{1}{\sigma}\frac{(1+s_\pi)(1+s_y)}{1-s_{kl}}\left(\omega^{\sigma-1}-1\right)\left[(1-\alpha)\frac{\sigma-1}{\sigma}\frac{1+s_y}{1-s_{kl}}\right]^{\frac{1-\alpha}{\alpha+\nu}}$. Computing the various derivatives, we get

$$\begin{aligned}
\frac{\partial H}{\partial K}&=\alpha\frac{1+\nu}{\alpha+\nu}\frac{1}{K}\left(H\left(K,\theta_{-1},\hat{v}\right)+f\right)\\
\frac{\partial H}{\partial\theta_{-1}}&=\frac{1+\nu}{\alpha+\nu}\frac{\rho\gamma_\theta}{\gamma_\theta+\gamma_v}\left(H\left(K,\theta_{-1},\hat{v}\right)+f\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial H}{\partial\hat{v}}&=\frac{1+\nu}{\alpha+\nu}\frac{\gamma_v}{\gamma_\theta+\gamma_v}\left(H\left(K,\theta_{-1},\hat{v}\right)+f\right)-H_0\left(Ae^{\frac{\gamma_\theta\rho\theta_{-1}+\gamma_v\hat{v}}{\gamma_\theta+\gamma_v}}K^\alpha\right)^{\frac{1+\nu}{\alpha+\nu}}\left(\frac{1+\nu}{\alpha+\nu}-\sigma+1\right)\frac{1}{\sigma-1}\frac{\sqrt{\gamma_v}\gamma_\theta}{\gamma_\theta+\gamma_v}\\
&\times\int e^{\frac{1+\nu}{\alpha+\nu}\varepsilon}\frac{\omega^{\sigma-1}-1}{\Omega^{\sigma-1}}\phi\left(\sqrt{\gamma_v}\left(\frac{\gamma_\theta}{\gamma_\theta+\gamma_v}\left(\rho\theta_{-1}-\hat{v}\right)+\varepsilon\right)\right)\Omega\left(\Phi\left(\sqrt{\gamma_v}\left(\frac{\gamma_\theta}{\gamma_\theta+\gamma_v}\left(\rho\theta_{-1}-\hat{v}\right)+\varepsilon\right)\right)\right)^{\frac{1+\nu}{\alpha+\nu}-\sigma+1}\\
&\times\sqrt{\gamma_\theta+\gamma_v}\phi\left(\sqrt{\gamma_\theta+\gamma_v}\varepsilon\right)d\varepsilon\\
&\geqslant\frac{1+\nu}{\alpha+\nu}\frac{\gamma_v}{\gamma_\theta+\gamma_v}\left(H\left(K,\theta_{-1},\hat{v}\right)+f\right)-\left(\frac{1+\nu}{\alpha+\nu}-\sigma+1\right)\frac{1}{\sqrt{2\pi}}\frac{\omega^{\sigma-1}-1}{\sigma-1}\frac{\sqrt{\gamma_v}\gamma_\theta}{\gamma_\theta+\gamma_v}\left(H\left(K,\theta_{-1},\hat{v}\right)+f\right)\\
&\geqslant\left(H\left(K,\theta_{-1},\hat{v}\right)+f\right)\frac{1+\nu}{\alpha+\nu}\frac{\sqrt{\gamma_v}}{\gamma_\theta+\gamma_v}\left(\sqrt{\gamma_v}-\frac{1}{\sqrt{2\pi}}\frac{\omega^{\sigma-1}-1}{\sigma-1}\gamma_\theta\right)\geqslant 0,
\end{aligned}$$

where the last inequality is a consequence of (18). Hence,

$$-\frac{\alpha\frac{1}{K}}{\frac{\sqrt{\gamma_v}}{\gamma_\theta+\gamma_v}\left(\sqrt{\gamma_v}-\frac{1}{\sqrt{2\pi}}\frac{\omega^{\sigma-1}-1}{\sigma-1}\gamma_\theta\right)}\leqslant\frac{\partial\hat{v}}{\partial K}\leqslant 0,$$

and

$$-\frac{\rho\gamma_\theta}{\sqrt{\gamma_v}\left(\sqrt{\gamma_v}-\frac{1}{\sqrt{2\pi}}\frac{\omega^{\sigma-1}-1}{\sigma-1}\gamma_\theta\right)}\leqslant\frac{\partial\hat{v}}{\partial\theta_{-1}}\leqslant 0,$$

which establishes the desired inequalities.

(ii) *Continuity and monotonicity of* m *and* $\overline{A}$. The continuity of m and $\overline{A}$ is inherited from that of $\hat{v}$ since

$$m\left(K,\boldsymbol{\theta}\right)=\Phi\left(\sqrt{\gamma_v}\left(\theta-\hat{v}\left(K,\theta_{-1}\right)\right)\right)\text{ and }\overline{A}\left(K,\boldsymbol{\theta}\right)=Ae^{\theta}\left(m\left(K,\boldsymbol{\theta}\right)\left(\omega^{\sigma-1}-1\right)+1\right)^{\frac{1}{\sigma-1}},$$

which are bounded on $\mathbb{K}\times\Theta^2$. The monotonicity of m and $\overline{A}$ is inherited from that of $\hat{v}$.

(iii) *Continuity and monotonicity of* $y\left(K,\boldsymbol{\theta}\right)$. Recall the definition of y:

$$y\left(K,\boldsymbol{\theta}\right)\equiv\left[\left(1-\alpha\right)\frac{\sigma-1}{\sigma}\frac{1+s_y}{1-s_{kl}}\right]^{\frac{1-\alpha}{\alpha+\nu}}\left(Ae^{\theta}\Omega\left(m\left(K,\boldsymbol{\theta}\right)\right)\right)^{\frac{1+\nu}{\alpha+\nu}}K^{\alpha\frac{1+\nu}{\alpha+\nu}}+\left(1-\delta\right)K-m\left(K,\boldsymbol{\theta}\right)f.$$

Computing the total derivative, we have:

$$\frac{dy}{dK}=\alpha\frac{1+\nu}{\alpha+\nu}\left[\left(1-\alpha\right)\frac{\sigma-1}{\sigma}\frac{1+s_y}{1-s_{kl}}\right]^{\frac{1-\alpha}{\alpha+\nu}}\left(Ae^{\theta}\Omega\left(m\left(K,\boldsymbol{\theta}\right)\right)\right)^{\frac{1+\nu}{\alpha+\nu}}K^{\alpha\frac{1+\nu}{\alpha+\nu}-1}+1-\delta+\frac{\partial y}{\partial m}\frac{\partial m}{\partial K}.$$

The first term and $\frac{\partial m}{\partial K}$ are positive, so we must only compute the sign of $\frac{\partial y}{\partial m}$. Compute the following:

$$\frac{\partial y}{\partial m}=\frac{1+\nu}{\alpha+\nu}\frac{1}{\sigma-1}\left[\left(1-\alpha\right)\frac{\sigma-1}{\sigma}\frac{1+s_y}{1-s_{kl}}\right]^{\frac{1-\alpha}{\alpha+\nu}}\left(Ae^{\theta}K^{\alpha}\right)^{\frac{1+\nu}{\alpha+\nu}}\left(\omega^{\sigma-1}-1\right)\Omega\left(m\left(K,\boldsymbol{\theta}\right)\right)^{\frac{1+\nu}{\alpha+\nu}-\sigma+1}-f.$$

Using the fact that

$$\begin{aligned}f&=\mathbb{E}_\theta\left[\frac{1}{\sigma}\frac{\left(1+s_\pi\right)\left(1+s_y\right)}{1-s_{kl}}\left(\omega^{\sigma-1}-1\right)\left[\left(1-\alpha\right)\frac{\sigma-1}{\sigma}\frac{1+s_y}{1-s_{kl}}\right]^{\frac{1-\alpha}{\alpha+\nu}}\left(Ae^{\theta}K^{\alpha}\right)^{\frac{1+\nu}{\alpha+\nu}}\Omega\left(m\right)^{\frac{1+\nu}{\alpha+\nu}-\sigma+1}\mid\theta_{-1},\hat{v}\right]\\&\leqslant\frac{1}{\sigma}\frac{\left(1+s_\pi\right)\left(1+s_y\right)}{1-s_{kl}}\left(\omega^{\sigma-1}-1\right)\left[\left(1-\alpha\right)\frac{\sigma-1}{\sigma}\frac{1+s_y}{1-s_{kl}}\right]^{\frac{1-\alpha}{\alpha+\nu}}\left(AK^{\alpha}\right)^{\frac{1+\nu}{\alpha+\nu}}\omega^{\frac{1+\nu}{\alpha+\nu}-\sigma+1}e^{\frac{\rho\gamma_\theta\theta_{-1}+\gamma_v\hat{v}}{\gamma_\theta+\gamma_v}+\frac{1}{2}\left(\frac{1+\nu}{\alpha+\nu}\right)^2\frac{1}{\gamma_\theta+\gamma_v}}.\end{aligned}$$

Consequently, we can bound $\frac{\partial y}{\partial m}$:

$$\left|\frac{\partial y}{\partial m}\right|\leqslant\left(\omega^{\sigma-1}-1\right)y_0,$$

where $y_0=\left[\left(1-\alpha\right)\frac{\sigma-1}{\sigma}\frac{1+s_y}{1-s_{kl}}\right]^{\frac{1-\alpha}{\alpha+\nu}}\left(AK^{\alpha}\right)^{\frac{1+\nu}{\alpha+\nu}}\omega^{\frac{1+\nu}{\alpha+\nu}-\sigma+1}\left(\frac{1+\nu}{\alpha+\nu}\frac{1}{\sigma-1}e^{\frac{1+\nu}{\alpha+\nu}\overline{\theta}}+\frac{1}{\sigma}\frac{\left(1+s_\pi\right)\left(1+s_y\right)}{1-s_{kl}}e^{\frac{\rho\gamma_\theta\theta_{-1}+\gamma_v\hat{v}}{\gamma_\theta+\gamma_v}+\frac{1}{2}\left(\frac{1+\nu}{\alpha+\nu}\right)^2\frac{1}{\gamma_\theta+\gamma_v}}\right)$.

Using our result from (i), we may then bound the following

$$\left|\frac{\partial y}{\partial m}\frac{\partial m}{\partial K}\right| \leqslant \left(\omega^{\sigma-1}-1\right) y_0 \frac{\alpha\frac{1}{K}}{\frac{\sqrt{\gamma_v}}{\gamma_\theta+\gamma_v}\left[\sqrt{\gamma_v}-\frac{1}{\sqrt{2\pi}}\frac{\omega^{\sigma-1}-1}{\sigma-1}\gamma_\theta\right]},$$

which means that function y is increasing in K for ω close enough to 1.
(iv) This property is used in several lemmas. The argument is the same as above:

$$y(K,\boldsymbol{\theta}) - \frac{L(K,\boldsymbol{\theta})^{1+\nu}}{1+\nu} = \left(1-\frac{\sigma-1}{\sigma}\frac{1-\alpha}{1+\nu}\frac{1+s_y}{1-s_{kl}}\right) y(K,\boldsymbol{\theta}) + (1-\delta) K - m(K,\boldsymbol{\theta}) f.$$

Therefore, under all the previous assumptions and assumption 1, then we can always find ω sufficiently close to 1 that $y(K,\boldsymbol{\theta}) - \frac{L(K,\boldsymbol{\theta})^{1+\nu}}{1+\nu}$ is increasing in K.
(v) *Monotonicity of R.* Recall the definition of R:

$$R(K,\theta_{-1},\theta) = R_0 \overline{A}(\theta, m(K,\theta_{-1},\theta))^{\frac{1+\nu}{\alpha+\nu}} K^{\alpha\frac{1+\nu}{\alpha+\nu}-1},$$

where $R_0 = \alpha\frac{\sigma-1}{\sigma}\frac{1+s_y}{1-s_{kl}}\left[(1-\alpha)\frac{\sigma-1}{\sigma}\frac{1+s_y}{1-s_{kl}}\right]^{\frac{1-\alpha}{\alpha+\nu}}$. Thus, the derivative with respect to K is

$$\frac{\partial R}{\partial K} = R(K,\theta_{-1},\theta)\left[-\left(1-\alpha\frac{1+\nu}{\alpha+\nu}\right)\frac{1}{K}+\frac{1+\nu}{\alpha+\nu}\frac{1}{\sigma-1}\frac{\omega^{\sigma-1}-1}{\Omega(m)^{\sigma-1}}\frac{\partial m}{\partial K}\right].$$

By definition $m(K,\theta_{-1},\theta) = \Phi\left(\sqrt{\gamma_v}\left(\theta-\hat{v}(K,\theta_{-1})\right)\right)$, we have

$$0 \leqslant \frac{\partial m}{\partial K} = -\sqrt{\gamma_v}\phi\left(\sqrt{\gamma_v}\left(\theta-\hat{v}(K,\theta_{-1})\right)\right)\frac{\partial \hat{v}}{\partial K} \leqslant \sqrt{\frac{\gamma_v}{2\pi}}\frac{\alpha\frac{1}{K}}{\frac{\sqrt{\gamma_v}}{\gamma_\theta+\gamma_v}\left(\sqrt{\gamma_v}-\frac{1}{\sqrt{2\pi}}\frac{\omega^{\sigma-1}-1}{\sigma-1}\gamma_\theta\right)}.$$

R is strictly increasing in K if

$$\left(1-\alpha\frac{1+\nu}{\alpha+\nu}\right)\frac{1}{K} > \frac{1+\nu}{\alpha+\nu}\frac{1}{\sigma-1}\frac{\omega^{\sigma-1}-1}{\Omega(m)^{\sigma-1}}\sqrt{\frac{\gamma_v}{2\pi}}\frac{\alpha\frac{1}{K}}{\frac{\sqrt{\gamma_v}}{\gamma_\theta+\gamma_v}\left(\sqrt{\gamma_v}-\frac{1}{\sqrt{2\pi}}\frac{\omega^{\sigma-1}-1}{\sigma-1}\gamma_\theta\right)}.$$

A simpler sufficient condition on parameters for the above condition to be satisfied is

$$\frac{1-\alpha\frac{1+\nu}{\alpha+\nu}}{\alpha\frac{1+\nu}{\alpha+\nu}} > \frac{1}{\sqrt{2\pi}}\frac{\omega^{\sigma-1}-1}{\sigma-1}\frac{\gamma_\theta+\gamma_v}{\sqrt{\gamma_v}-\frac{1}{\sqrt{2\pi}}\frac{\omega^{\sigma-1}-1}{\sigma-1}\gamma_\theta}.$$

□

F.4 Existence and Uniqueness of the Dynamic Equilibrium

This proof builds on the monotone operator and lattice-theoretic techniques developed in Coleman (1991), Coleman and John (2000), Datta et al. (2002) or Morand and Reffett (2003) and extends it to the features present in our setup. The proof uses the following version of Tarski's fixed point theorem (see Tarski et al. (1955)):

Theorem. ***[Tarski, 1955]*** *Suppose that $(X, \geqslant)$ is a nonempty complete lattice and $T : X \to X$ is an increasing mapping. Then, the set of fixed points of T is a nonempty complete lattice.*

Description and Definitions

The objective of this proof is to show the existence and uniqueness of a solution to the Euler equation in some particular space. For reasons that will appear clearer later, it is useful to represent the Euler equation in the space of *inverse marginal utility*, which we denote as p, instead of consumption functions directly.[36] That is to say, we will go back and forth between the spaces of inverse marginal values and consumption functions through the following mapping,

$$p\left(K, \boldsymbol{\theta}\right) = U_C\left(c\left(K, \boldsymbol{\theta}\right), L\left(K, \boldsymbol{\theta}\right)\right)^{-1}.$$

Definition 5. Let $\mathscr{P}$ be the set

$$\begin{aligned}\mathscr{P} = \Big\{ & p\left(K, \boldsymbol{\theta}\right) \mid p : \mathbb{K} \times \Theta^2 \longrightarrow \mathbb{K} \text{ such that} \\ & (a)\, 0 \leqslant p\left(K, \boldsymbol{\theta}\right) \leqslant U_C\left(y\left(K, \boldsymbol{\theta}\right), L\left(K, \boldsymbol{\theta}\right)\right)^{-1} \text{ for } \left(K, \boldsymbol{\theta}\right) \in \mathbb{K} \times \Theta^2; \\ & (b)\, p \text{ weakly increasing in } K \Big\}.\end{aligned}$$

Definition 5 describes the set in which the equilibrium inverse marginal utility p lies. We may now introduce the following definitions which sets up the environment and the Euler equation that we must solve:

Definition 6. (i) The mapping from marginal consumption value to consumption is

$$\begin{aligned} & C : \mathbb{R}^+ \times \mathbb{K} \times \Theta^2 \longrightarrow \mathbb{R} \\ & \left(p, K, \boldsymbol{\theta}\right) \mapsto U_C^{-1}\left(p, L\left(K, \boldsymbol{\theta}\right)\right) = p^{\frac{1}{\gamma}} + \frac{L\left(K, \boldsymbol{\theta}\right)^{1+\nu}}{1+\nu};\end{aligned}$$

(ii) The mapping corresponding to the Euler equation is

$$\begin{aligned} & Z : \mathbb{R}^+ \times \mathscr{P} \times \mathbb{K} \times \Theta^2 \longrightarrow \mathbb{R} \cup \{-\infty, \infty\} \\ & \left(p, P, K, \boldsymbol{\theta}\right) \mapsto \begin{cases} 0 \text{ if } p = 0 \text{ and } \Big(K = 0 \text{ or } P\Big(y\left(K, \boldsymbol{\theta}\right) - C\left(0, K, \boldsymbol{\theta}\right), \boldsymbol{\theta}'\Big) = 0\Big) \\ \frac{1}{p} - \beta \mathbb{E}\left[\frac{r\Big(y(K,\boldsymbol{\theta}) - C(p,K,\boldsymbol{\theta}), \boldsymbol{\theta}'\Big)}{P\Big(y(K,\boldsymbol{\theta}) - C(p,K,\boldsymbol{\theta}), \boldsymbol{\theta}'\Big)}\right], \text{ otherwise;} \end{cases}\end{aligned}$$

[36] A similar existence proof can be written in the space of consumption functions as in Coleman (1991). The uniqueness is, however, problematic in that space since the operator corresponding to the Euler equation is not pseudo-concave without restrictive assumptions on the preferences. On the other hand, that same operator is naturally pseudo-concave in the space of inverse marginal utilities as noted by Coleman (2000) and Datta et al. (2002).

(iii) The operator providing solutions to the Euler equation is

$$T(P) = \left\{p \in \mathscr{P} \mid Z(p(K,\boldsymbol{\theta}), P, K, \boldsymbol{\theta}) = 0 \text{ for } K \in \mathbb{K}, \boldsymbol{\theta} \in \Theta^2\right\}.$$

Existence

We endow the space $\mathscr{P}$ with the pointwise partial order $\leqslant$, such that $p \leqslant \hat{p}$ if $p(K,\boldsymbol{\theta}) \leqslant \hat{p}(K,\boldsymbol{\theta})$ for all $(K,\boldsymbol{\theta}) \in \mathbb{K} \times \Theta^2$ and two binary operations that we refer to as the *meet* $(p \wedge \hat{p})$ and the *join* $(p \vee \hat{p})$ for any two points $p, \hat{p} \in \mathscr{P}$. The meet is the greatest lower bound of two elements, i.e.,

$$(p \wedge \hat{p})(K,\boldsymbol{\theta}) = \min\ \{p(K,\boldsymbol{\theta}), \hat{p}(K,\boldsymbol{\theta})\},$$

and the join is the least upper bound, defined as

$$(p \vee \hat{p})(K,\boldsymbol{\theta}) = \max\ \{p(K,\boldsymbol{\theta}), \hat{p}(K,\boldsymbol{\theta})\}.$$

Lemma A3. *$(\mathscr{P}, \leqslant)$ is a complete lattice.*

Proof. A lattice is complete if each subset has a supremum and an infimum. Consider a subset $X \subset \mathscr{P}$. Clearly, the join of all elements in X, $\sup\limits_{p\in X} p$, satisfies $\sup\limits_{p\in X} p \leqslant U_C(y(K,\boldsymbol{\theta}), L(K,\boldsymbol{\theta}))^{-1}$ and $\sup\limits_{p\in X} p$ is weakly increasing in K, so $\sup\limits_{p\in X} p \in \mathscr{P}$. A symmetric argument tells us that the meet of all elements in X, $\inf\limits_{p\in X} p$, belongs to $\mathscr{P}$. Therefore, $\mathscr{P}$ is a complete lattice. □

We now show that mapping T, which associates the solution to the Euler equation for any future inverse marginal utility $P \in \mathscr{P}$ is a well-defined monotone mapping from $\mathscr{P}$ to $\mathscr{P}$.

Lemma A4. *Under the conditions of proposition A1, assumption 1 and ω close enough to 1 such that $y(K,\boldsymbol{\theta}) - \frac{L(K,\boldsymbol{\theta})^{1+\nu}}{1+\nu}$ is nondecreasing in K (lemma A2.(iv)) and R is nonincreasing in K (lemma A2.(v)) , T is a well-defined self-map on $\mathscr{P}$.*

Proof. Notice, first, from the definition of Z and $\mathscr{P}$ that Z is strictly decreasing in p but strictly increasing in P.

Step 1: T is well defined. Fix $K > 0$, $\boldsymbol{\theta}$ and P. Note that as $p \to 0$, $Z(p, P, K, \boldsymbol{\theta}) \to \infty$ and as $p \to U_C(y(K,\boldsymbol{\theta}), L(K,\boldsymbol{\theta}))^{-1}$, $Z(p, P, K, \boldsymbol{\theta}) \to -\infty$. For ω close enough to 1, r in nonincreasing in K. Thus, Z is continuous and strictly decreasing in p, there exists a unique $0 < p(K,\boldsymbol{\theta}) < Y(K,\boldsymbol{\theta})$ such that $Z(p(K,\boldsymbol{\theta}), P, K, \boldsymbol{\theta}) = 0$.

Step 2: T maps $\mathscr{P}$ onto itself. We must check properties (a)-(b) in the definition of $\mathscr{P}$:

(a) Already verified in step 1.

(b) Pick $0 < K \leqslant \hat{K}$. Denote $p = T(P)$. By definition $Z(p(K,\boldsymbol{\theta}), P, K, \boldsymbol{\theta}) = 0$. Evaluate Z at

$p(K,\boldsymbol{\theta})$ for $\hat{K}$:

$$Z\left(p(K,\boldsymbol{\theta}),P,\hat{K},\boldsymbol{\theta}\right)=\frac{1}{p(K,\boldsymbol{\theta})}-\beta\mathbb{E}\left[\frac{r\Big(y\left(\hat{K},\boldsymbol{\theta}\right)-C\left(p(K,\boldsymbol{\theta}),\hat{K},\boldsymbol{\theta}\right),\boldsymbol{\theta}'\Big)}{P\Big(y\left(\hat{K},\boldsymbol{\theta}\right)-C\left(p(K,\boldsymbol{\theta}),\hat{K},\boldsymbol{\theta}\right),\boldsymbol{\theta}'\Big)}\right].$$

Compute the following term:

$$\begin{aligned}
y\left(\hat{K},\boldsymbol{\theta}\right)-C\left(p(K,\boldsymbol{\theta}),\hat{K},\boldsymbol{\theta}\right)&=y\left(\hat{K},\boldsymbol{\theta}\right)-\frac{L\left(\hat{K},\boldsymbol{\theta}\right)^{1+\nu}}{1+\nu}-p(K,\boldsymbol{\theta})^{\frac{1}{\gamma}}\\
&=\left(1-\frac{\sigma-1}{\sigma}\frac{1-\alpha}{1+\nu}\frac{1+s_y}{1-s_{kl}}\right)y\left(\hat{K},\boldsymbol{\theta}\right)-m\left(\hat{K},\boldsymbol{\theta}\right)f+(1-\delta)\hat{K}-p(K,\boldsymbol{\theta})^{\frac{1}{\gamma}}\\
&\geqslant y(K,\boldsymbol{\theta})-C\left(p(K,\boldsymbol{\theta}),K,\boldsymbol{\theta}\right),
\end{aligned}$$

where the inequality is due to the fact that $y-\frac{L^{1+\nu}}{1+\nu}$ is increasing in K for ω close enough to 1 (lemma A2(iv)). Therefore,

$$Z\left(p(K,\boldsymbol{\theta}),P,\hat{K},\boldsymbol{\theta}\right)\geqslant Z\left(p(K,\boldsymbol{\theta}),P,K,\boldsymbol{\theta}\right)=0,$$

which implies that $p\left(\hat{K},\boldsymbol{\theta}\right)\geqslant p(K,\boldsymbol{\theta})$ since Z is strictly decreasing in p. □

Lemma A5. *Under the conditions of lemma A4, T is continuous and monotone.*

Proof. Step 1: Monotonicity.

Take $p\leqslant\hat{p}$ in the sense that $p(K,\boldsymbol{\theta})\leqslant\hat{p}(K,\boldsymbol{\theta})$ for all $(K,\boldsymbol{\theta})$. Evaluate Z at

$$\begin{aligned}
Z\left(Tp(K,\boldsymbol{\theta}),\hat{p},K,\boldsymbol{\theta}\right)&=\frac{1}{Tp(K,\boldsymbol{\theta})}-\beta\mathbb{E}\left[\frac{r\Big(y(K,\boldsymbol{\theta})-C\left(Tp(K,\boldsymbol{\theta}),\hat{K},\boldsymbol{\theta}\right),\boldsymbol{\theta}'\Big)}{\hat{p}\Big(y(K,\boldsymbol{\theta})-C\left(Tp(K,\boldsymbol{\theta}),\hat{K},\boldsymbol{\theta}\right),\boldsymbol{\theta}'\Big)}\right]\\
&\geqslant Z\left(Tp(K,\boldsymbol{\theta}),p,K,\boldsymbol{\theta}\right)=0,
\end{aligned}$$

which implies that $T\hat{p}(K,\boldsymbol{\theta})\geqslant Tp(K,\boldsymbol{\theta})$. Therefore, $Tp\leqslant T\hat{p}$.

Step 2: Continuity.

Fix $p\in\mathscr{P}$. Pick $\varepsilon>0$ and some $\hat{p}\in\mathscr{P}$ such that $\|\hat{p}-p\|\leqslant\varepsilon$. Fix $K>0,\boldsymbol{\theta}\in\Theta^2$. For all $\tilde{p}\in\mathbb{R}$,

$$\begin{aligned}
Z\left(\tilde{p},\hat{p},K,\boldsymbol{\theta}\right)&=\frac{1}{\tilde{p}}-\beta\mathbb{E}\left[\frac{r\Big(y(K,\boldsymbol{\theta})-C\left(\tilde{p},\hat{K},\boldsymbol{\theta}\right),\boldsymbol{\theta}'\Big)}{\hat{p}\Big(y(K,\boldsymbol{\theta})-C\left(\tilde{p},\hat{K},\boldsymbol{\theta}\right),\boldsymbol{\theta}'\Big)}\right]\\
&\leqslant\frac{1}{\tilde{p}}-\beta\mathbb{E}\left[\frac{r\Big(y(K,\boldsymbol{\theta})-C\left(\tilde{p},\hat{K},\boldsymbol{\theta}\right),\boldsymbol{\theta}'\Big)}{p\Big(y(K,\boldsymbol{\theta})-C\left(\tilde{p},\hat{K},\boldsymbol{\theta}\right),\boldsymbol{\theta}'\Big)+\varepsilon}\right],
\end{aligned}$$

which means that $T\hat{p} \leqslant T(p+\varepsilon)$. A similar argument yields $T\hat{p} \geqslant T(p-\varepsilon)$. By definition,

$$Z(T[p+\varepsilon](K,\boldsymbol{\theta}), p+\varepsilon, K, \boldsymbol{\theta}) = \frac{1}{T[p+\varepsilon](K,\boldsymbol{\theta})} - \beta\mathbb{E}\left[\frac{r\Big(y(K,\boldsymbol{\theta}) - C\Big(T[p+\varepsilon](K,\boldsymbol{\theta}), \hat{K}, \boldsymbol{\theta}\Big), \boldsymbol{\theta}'\Big)}{p\Big(y(K,\boldsymbol{\theta}) - C\Big(T[p+\varepsilon](K,\boldsymbol{\theta}), \hat{K}, \boldsymbol{\theta}\Big), \boldsymbol{\theta}'\Big) + \varepsilon}\right].$$

Using the fact that $T(p+\varepsilon) \geqslant Tp$ and r/p is decreasing in K, we obtain:

$$0 \leqslant \frac{1}{T[p+\varepsilon](K,\boldsymbol{\theta})} - \beta\mathbb{E}\left[\frac{r\Big(y(K,\boldsymbol{\theta}) - C\Big(Tp(K,\boldsymbol{\theta}), \hat{K}, \boldsymbol{\theta}\Big), \boldsymbol{\theta}'\Big)}{p\Big(y(K,\boldsymbol{\theta}) - C\Big(Tp(K,\boldsymbol{\theta}), \hat{K}, \boldsymbol{\theta}\Big), \boldsymbol{\theta}'\Big) + \varepsilon}\right].$$

Thus,

$$\begin{aligned}
T[p+\varepsilon](K,\boldsymbol{\theta}) &\leqslant \beta^{-1}\mathbb{E}\left[\frac{r\Big(y(K,\boldsymbol{\theta}) - C\Big(Tp(K,\boldsymbol{\theta}), \hat{K}, \boldsymbol{\theta}\Big), \boldsymbol{\theta}'\Big)}{p\Big(y(K,\boldsymbol{\theta}) - C\Big(Tp(K,\boldsymbol{\theta}), \hat{K}, \boldsymbol{\theta}\Big), \boldsymbol{\theta}'\Big) + \varepsilon}\right]^{-1} \\
&\leqslant \beta^{-1}\mathbb{E}\left[\frac{p\Big(y(K,\boldsymbol{\theta}) - C\Big(Tp(K,\boldsymbol{\theta}), \hat{K}, \boldsymbol{\theta}\Big), \boldsymbol{\theta}'\Big) + \varepsilon}{r\Big(y(K,\boldsymbol{\theta}) - C\Big(Tp(K,\boldsymbol{\theta}), \hat{K}, \boldsymbol{\theta}\Big), \boldsymbol{\theta}'\Big)}\right] \quad \text{(Jensen)} \\
&\leqslant Tp(K,\boldsymbol{\theta}) + \beta^{-1}\varepsilon\mathbb{E}\left[r\Big(y(K,\boldsymbol{\theta}) - C\Big(Tp(K,\boldsymbol{\theta}), \hat{K}, \boldsymbol{\theta}\Big), \boldsymbol{\theta}'\Big)^{-1}\right] \\
&\leqslant Tp(K,\boldsymbol{\theta}) + \beta^{-1} r\Big(y\left(\overline{K}, \overline{\boldsymbol{\theta}}\right), \overline{\boldsymbol{\theta}}\Big)^{-1}\varepsilon.
\end{aligned}$$

The same argument applied to $p-\varepsilon$ yields $Tp \leqslant T(p-\varepsilon) + \beta^{-1} r\Big(y\left(\overline{K}, \overline{\boldsymbol{\theta}}\right), \overline{\boldsymbol{\theta}}\Big)^{-1}\varepsilon$. We can now conclude that T is a continuous mapping on $\mathscr{P}$, since $\|\hat{p} - p\| \leqslant \varepsilon$ implies

$$\begin{aligned}
\|T\hat{p} - Tp\| &\leqslant \max\left(\|T(p+\varepsilon) - Tp\|, \|T(p-\varepsilon) - Tp\|\right) \\
&\leqslant \beta^{-1} r\Big(y\left(\overline{K}, \overline{\boldsymbol{\theta}}\right), \overline{\boldsymbol{\theta}}\Big)^{-1}\varepsilon.
\end{aligned}$$

□

Proposition A2. *Under the conditions of lemma A4 and assumption 2, there exists a strictly positive equilibrium function $p^* \in \mathscr{P}$.*

Proof. The existence of a fixed point is simply given by Tarski's fixed point theorem applied to the monotone self-map T on the complete lattice $(\mathscr{P}, \leqslant)$.

We now construct a strictly positive fixed point p^*. Note that we are abusing language when using the expression "strictly positive", since our setup is such that $p^*(0,\boldsymbol{\theta}) = 0$ for all $\boldsymbol{\theta}$. Thus, by "strictly positive", we mean that $p^*(K,\boldsymbol{\theta}) > 0$ for all $K > 0$. We proceed in three steps.

Step 1. Define the sequence $(p_n)_{n\geqslant 0}$ such that $p_0(K,\boldsymbol{\theta}) = U_C(y(K,\boldsymbol{\theta}), L(K,\boldsymbol{\theta}))^{-1}$ and $p_n = T^n p_0$. By construction, the first iteration is mapped downward ($p_1 \leqslant p_0$) and we obtain a decreasing sequence which converges pointwise towards a function p^*. Clearly, $p^* = \inf_{n\geqslant 0} p_n$ so $p^* \in \mathscr{P}$. Furthermore, since T is continuous, $p^* = Tp^*$, so p^* is a fixed point of T.

Step 2. We first show that p^* is not 0. From assumption 2, take K^-such that $y\left(K^-,\underline{\boldsymbol{\theta}}\right) - \frac{L\left(K^-,\underline{\boldsymbol{\theta}}\right)^{1+\nu}}{1+\nu} > K^-$ and $\beta\mathbb{E}\left[r\left(K^-,\underline{\theta},\theta'\right) \mid \underline{\theta}\right] \leqslant 1$. Pick an $\alpha > 0$ such that $C\left(\alpha,K^-,\underline{\boldsymbol{\theta}}\right) < y\left(K^-,\underline{\boldsymbol{\theta}}\right) - K^-$, i.e., such that $0 < \alpha^{\frac{1}{\gamma}} < y\left(K^-,\underline{\boldsymbol{\theta}}\right) - \frac{L\left(K^-,\underline{\boldsymbol{\theta}}\right)^{1+\nu}}{1+\nu} - K^-$. Assume some $p \in \mathscr{P}$ is such that $p\left(K^-,\underline{\theta},\theta'\right) \geqslant \alpha$ for all θ', then we show that $Tp\left(K^-,\underline{\boldsymbol{\theta}}\right) \geqslant \alpha$ by simply evaluating $Z\left(\alpha,p,K^-,\underline{\boldsymbol{\theta}}\right)$:

$$
\begin{aligned}
Z\left(\alpha,p,K^-,\underline{\boldsymbol{\theta}}\right) &= \frac{1}{\alpha} - \beta\mathbb{E}\left[\frac{r\Big(y\left(K^-,\underline{\boldsymbol{\theta}}\right) - C\left(\alpha,K^-,\underline{\boldsymbol{\theta}}\right),\underline{\theta},\theta'\Big)}{p\Big(y\left(K^-,\underline{\boldsymbol{\theta}}\right) - C\left(\alpha,K^-,\underline{\boldsymbol{\theta}}\right),\underline{\theta},\theta'\Big)}\right] \\
&\geqslant \frac{1}{\alpha} - \beta\mathbb{E}\left[\frac{r\Big(K^-,\underline{\theta},\theta'\Big)}{p\Big(K^-,\underline{\theta},\theta'\Big)}\right] \geqslant \frac{1}{\alpha} - \frac{\beta}{\alpha}\mathbb{E}\left[r\Big(K^-,\underline{\theta},\theta'\Big)\right] \geqslant 0.
\end{aligned}
$$

This establishes that $Tp\left(K^-,\boldsymbol{\theta}\right) \geqslant \alpha$. Since we start our iterations with

$$
p_0\left(K^-,\boldsymbol{\theta}\right) = U_C\left(y\left(K^-,\boldsymbol{\theta}\right),L\left(K^-,\boldsymbol{\theta}\right)\right)^{-1},
$$

i.e., such that $C\left(p_0,K^-,\underline{\theta},\theta'\right) = y\left(K^-,\underline{\theta},\theta'\right)$ and therefore $p_0\left(K^-,\underline{\theta},\theta'\right) > \alpha$, we have $p^*\left(K^-,\underline{\theta},\theta'\right) \geqslant \alpha > 0$.

Step 3. We now want to show that p^* is strictly positive for all $K > 0$. Assume, by contradiction, that p^* is not strictly positive. This means, that there exists $\left(K_0,\boldsymbol{\theta}_0\right)$ such that $p^*\left(K_0,\boldsymbol{\theta}_0\right) = 0$. Since p^* is increasing in all its arguments, this means that $p^*\left(K,\underline{\boldsymbol{\theta}}\right) = 0$ for all $K \leqslant K_0$. Define

$$
\tilde{K} = \sup_{K \leqslant K^-} \left\{p^*\left(K,\underline{\boldsymbol{\theta}}\right) = 0\right\}.
$$

With the assumption that p^* is not strictly positive, $\tilde{K} > 0$. Since $\tilde{K} \leqslant K^-$, then we have that $y\left(\tilde{K},\underline{\boldsymbol{\theta}}\right) - \frac{L\left(\tilde{K},\underline{\boldsymbol{\theta}}\right)^{1+\nu}}{1+\nu} > \tilde{K}$. The right hand side of the Euler equation evaluated at $\tilde{K}$ and $\underline{\boldsymbol{\theta}}$ gives

$$
0 \leqslant \beta\mathbb{E}\left[\frac{r\Big(y\left(\tilde{K},\underline{\boldsymbol{\theta}}\right) - C\left(0,\tilde{K},\underline{\boldsymbol{\theta}}\right),\boldsymbol{\theta}'\Big)}{p^*\Big(y\left(\tilde{K},\underline{\boldsymbol{\theta}}\right) - C\left(0,\tilde{K},\underline{\boldsymbol{\theta}}\right),\boldsymbol{\theta}'\Big)}\right] \leqslant \beta\mathbb{E}\left[\frac{r\Big(\tilde{K},\boldsymbol{\theta}'\Big)}{p^*\Big(y\left(\tilde{K},\underline{\boldsymbol{\theta}}\right) - C\left(0,\tilde{K},\underline{\boldsymbol{\theta}}\right),\boldsymbol{\theta}'\Big)}\right],
$$

which is finite since $p^*\Big(y\left(\tilde{K},\underline{\boldsymbol{\theta}}\right) - C\left(0,\tilde{K},\underline{\boldsymbol{\theta}}\right),\boldsymbol{\theta}'\Big) > 0$. Thus, we obtain a contradiction since $p^*\left(\tilde{K},\underline{\theta}\right) = 0$ cannot be a solution. Therefore, $\tilde{K}$ must be 0 and p^* is strictly positive everywhere except at $K = 0$. □

Uniqueness

The proof for uniqueness relies on showing that the operator T is pseudo-concave. The following definitions are useful for that purpose. First, define a pseudo-concave operator:

Definition 7. A monotone operator $T:\mathscr{P}\longrightarrow\mathscr{P}$ is pseudo-concave if for any strictly positive $p\in\mathscr{P}$ and $t\in(0,1)$, $T(tp)(K,\boldsymbol{\theta})>tTp(K,\boldsymbol{\theta})$ for all $K>0$, $\boldsymbol{\theta}\in\Theta^2$.

We now define the concept of K_0-monotonicity for an operator.

Definition 8. An operator $T:\mathscr{P}\longrightarrow\mathscr{P}$ is K_0-monotone if it is monotone and if, for any strictly positive fixed point p^*, there exists $K_0>0$ such that for any $0\leqslant K_1\leqslant K_0$ and any $p\in\mathscr{P}$ such that $p(K,\boldsymbol{\theta})\leqslant p^*(K,\boldsymbol{\theta}),\forall K\geqslant K_1,\boldsymbol{\theta}$, then

$$p^*(K,\boldsymbol{\theta})\geqslant Tp(K,\boldsymbol{\theta}),\forall K\geqslant K_1,\boldsymbol{\theta}.$$

We now proceed to show that T is K_0-monotone, which we will then use to prove its pseudo-concavity. In order to do so, the following preliminary result is useful:

Lemma A6. *Under the conditions of lemma A4, suppose $P\in\mathscr{P}$ and let $p=T(P)$, then for all $\left(\theta_{-1},\hat{\theta}_{-1}\right)\in\Theta^2$,*

$$\left|C\left(p\left(K,\hat{\theta}_{-1},\theta\right),K,\hat{\theta}_{-1},\theta\right)-C\left(p\left(K,\theta_{-1},\theta\right),K,\theta_{-1},\theta\right)\right|\leqslant\left|y\left(K,\hat{\theta}_{-1},\theta\right)-y\left(K,\theta_{-1},\theta\right)\right|.$$

Proof. Pick $\left(\theta_{-1},\hat{\theta}_{-1}\right)\in\Theta^2$ and assume WLOG that $Y\left(K,\hat{\theta}_{-1},\theta\right)\geqslant Y\left(K,\theta_{-1},\theta\right)$ and that ω has been chosen close enough to 1 that

$$y\left(K,\hat{\theta}_{-1},\theta\right)-\frac{L\left(K,\hat{\theta}_{-1},\theta\right)^{1+\nu}}{1+\nu}\geqslant y\left(K,\theta_{-1},\theta\right)-\frac{L\left(K,\theta_{-1},\theta\right)^{1+\nu}}{1+\nu}.$$

Step 1. By definition $Z\left(p\left(K,\theta_{-1},\theta\right),P,K,\theta_{-1},\theta\right)=0$. Evaluate $Z\left(\tilde{p},P,K,\hat{\theta}_{-1},\theta\right)$ at $\tilde{p}$ such that

$$C\left(\tilde{p},P,K,\hat{\theta}_{-1},\theta\right)=C\left(p\left(K,\theta_{-1},\theta\right),K,\theta_{-1},\theta\right)+y\left(K,\hat{\theta}_{-1},\theta\right)-y\left(K,\theta_{-1},\theta\right),$$

in other words,

$$\tilde{p}^{\frac{1}{\gamma}}+\frac{L\left(K,\hat{\theta}_{-1},\theta\right)^{1+\nu}}{1+\nu}=p^{\frac{1}{\gamma}}+\frac{L\left(K,\theta_{-1},\theta\right)^{1+\nu}}{1+\nu}+y\left(K,\hat{\theta}_{-1},\theta\right)-y\left(K,\theta_{-1},\theta\right).$$

Assume WLOG that $\tilde{p}\geqslant p\left(K,\theta_{-1},\theta\right)$. Then, we have

$$\begin{aligned}
Z\left(\tilde{p},P,K,\hat{\theta}_{-1},\theta\right)&=\frac{1}{\tilde{p}}-\beta\mathbb{E}\left[\frac{r\Big(y\left(K,\hat{\theta}_{-1},\theta\right)-C\left(\tilde{p},K,\hat{\theta}_{-1},\theta\right),\theta,\theta'\Big)}{P\Big(y\left(K,\hat{\theta}_{-1},\theta\right)-C\left(\tilde{p},K,\hat{\theta}_{-1},\theta\right),\theta,\theta'\Big)}\right]\\
&=\frac{1}{\tilde{p}}-\beta\mathbb{E}\left[\frac{r\Big(y\left(K,\theta_{-1},\theta\right)-C\left(\tilde{p},K,\theta_{-1},\theta\right),\theta,\theta'\Big)}{P\Big(y\left(K,\theta_{-1},\theta\right)-C\left(\tilde{p},K,\theta_{-1},\theta\right),\theta,\theta'\Big)}\right]\\
&\leqslant 0,
\end{aligned}$$

which tells us that $p\left(K,\hat{\theta}_{-1},\theta\right)\leqslant\tilde{p}$. Thus, we have:

$$
\begin{aligned}
&C\left(p\left(K,\hat{\theta}_{-1},\theta\right),K,\hat{\theta}_{-1},\theta\right)-C\left(p\left(K,\theta_{-1},\theta\right),K,\theta_{-1},\theta\right)\\
&\leqslant C\left(\tilde{p},K,\hat{\theta}_{-1},\theta\right)-C\left(p\left(K,\theta_{-1},\theta\right),K,\theta_{-1},\theta\right)\\
&\leqslant y\left(K,\hat{\theta}_{-1},\theta\right)-y\left(K,\theta_{-1},\theta\right).
\end{aligned}
$$

Step 2. We now evaluate the other side of the inequality. Evaluate $Z\left(p\left(K,\theta_{-1},\theta\right),P,K,\hat{\theta}_{-1},\theta\right)$:

$$
\begin{aligned}
Z\left(p\left(K,\theta_{-1},\theta\right),P,K,\hat{\theta}_{-1},\theta\right)&=\frac{1}{p\left(K,\theta_{-1},\theta\right)}-\beta\mathbb{E}\left[\frac{r\Big(y\left(K,\hat{\theta}_{-1},\theta\right)-C\left(p\left(K,\theta_{-1},\theta\right),K,\hat{\theta}_{-1},\theta\right),\theta,\theta'\Big)}{P\Big(y\left(K,\hat{\theta}_{-1},\theta\right)-C\left(p\left(K,\theta_{-1},\theta\right),K,\hat{\theta}_{-1},\theta\right),\theta,\theta'\Big)}\right]\\
&\geqslant\frac{1}{p\left(K,\theta_{-1},\theta\right)}-\beta\mathbb{E}\left[\frac{r\Big(y\left(K,\theta_{-1},\theta\right)-C\left(p\left(K,\theta_{-1},\theta\right),K,\theta_{-1},\theta\right),\theta,\theta'\Big)}{P\Big(y\left(K,\theta_{-1},\theta\right)-C\left(p\left(K,\theta_{-1},\theta\right),K,\theta_{-1},\theta\right),\theta,\theta'\Big)}\right],
\end{aligned}
$$

which implies that $p\left(K,\hat{\theta}_{-1},\theta\right)\geqslant p\left(K,\theta_{-1},\theta\right)$. Therefore,

$$
C\left(p\left(K,\hat{\theta}_{-1},\theta\right),K,\hat{\theta}_{-1},\theta\right)-C\left(p\left(K,\theta_{-1},\theta\right),K,\theta_{-1},\theta\right)\geqslant 0,
$$

which establishes the desired result. □

Lemma A7. *Under the conditions of lemma A4 and assumption 2, T is K_0-monotone.*

Proof. The proof proceeds in two steps.
Step 1. Let us show that there exists $K_0>0$ such that $y\left(K,\boldsymbol{\theta}\right)-C\left(p^*\left(K,\boldsymbol{\theta}\right),K,\boldsymbol{\theta}\right)\geqslant K,\forall K\leqslant K_0,\forall\boldsymbol{\theta}$. Pick a strictly positive fixed point p^*. By contradiction, suppose that for all $K_0>0$, there exists a $K\leqslant K_0$ and a $\boldsymbol{\theta}=\left(\theta_{-1},\theta\right)'$ such that $y\left(K,\boldsymbol{\theta}\right)-C\left(p^*\left(K,\boldsymbol{\theta}\right),K,\boldsymbol{\theta}\right)<K$. Suppose, by contradiction, that $y\left(K,\theta,\theta\right)-C\left(p^*\left(K,\theta,\theta\right),K,\theta,\theta\right)\geqslant K$, then we would have

$$
C\left(p^*\left(K,\boldsymbol{\theta}\right),K,\boldsymbol{\theta}\right)-C\left(p^*\left(K,\theta,\theta\right),K,\theta,\theta\right)>y\left(K,\theta_{-1},\theta\right)-y\left(K,\theta,\theta\right),
$$

which cannot be true for $p^*\in\mathscr{P}$ according to lemma A6. Therefore, $y\left(K,\theta,\theta\right)-C\left(p^*\left(K,\theta,\theta\right),K,\theta,\theta\right)<K$. By the definition of p^*:

$$
\begin{aligned}
\frac{1}{p^*\left(K,\theta,\theta\right)}&=\beta\mathbb{E}\left[\frac{r\left(y\left(K,\theta,\theta\right)-C\left(p^*\left(K,\theta,\theta\right),K,\theta,\theta\right),\theta,\theta'\right)}{p^*\left(y\left(K,\theta,\theta\right)-C\left(p^*\left(K,\theta,\theta\right),K,\theta,\theta\right),\theta,\theta'\right)}\right]\\
&>\beta\mathbb{E}\left[\frac{r\left(K,\theta,\theta'\right)}{p^*\left(K,\theta,\theta'\right)}\right]=\beta\int\frac{r\left(K,\theta,\theta'\right)}{p^*\left(K,\theta,\theta'\right)}\pi\left(d\theta',\theta\right),
\end{aligned}
$$

where $\pi\left(d\theta',\theta\right)$ denotes the marginal density of θ' conditional on θ. Since $p^*\left(K,\theta,\theta'\right)$ is weakly increasing in θ', we have

$$\begin{aligned}
&\beta \int \frac{r(K,\theta,\theta')}{p^*(K,\theta,\theta')} \pi\left(d\theta',\theta\right) \\
&\geqslant \frac{\beta}{p^*(K,\theta,\theta)} \int_{\theta'\leqslant\theta} r\left(K,\theta,\theta'\right) \pi\left(d\theta',\theta\right) \\
&= \frac{\beta}{p^*(K,\theta,\theta)} r\left(K,\underline{\theta},\underline{\theta}\right) \mathbb{P}\left(\theta' \leqslant \theta\right),
\end{aligned}$$

where $r(K,\underline{\theta},\underline{\theta}) > 0$. Given our specification of the stochastic process followed by θ, denote $\underline{\pi}$ the lower bound on the probability that θ' falls below current θ, i.e., $\underline{\pi} = \inf_{\theta\in\Theta} \mathbb{P}(\theta' \leqslant \theta \mid \theta)$. In our setting, $\underline{\pi}$ exists and is strictly positive. Since $r(K,\underline{\theta},\underline{\theta}) \to \infty$ as $K \to 0$, we can choose K_0 small enough that $\beta r(K,\underline{\theta},\underline{\theta})\,\underline{\pi} > 1$, then

$$\begin{aligned}
\frac{1}{p^*(K,\theta,\theta)} &> \frac{\beta}{p^*(K,\theta,\theta)} r\left(K,\underline{\theta},\underline{\theta}\right)\underline{\pi}, \\
&> \frac{1}{p^*(K,\theta,\theta)}.
\end{aligned}$$

Hence, we have a contradiction.

Step 2. Keeping the same K_0 given by step 1, pick a $K_1 \leqslant K_0$ with a p such that $p(K,\boldsymbol{\theta}) \leqslant p^*(K,\boldsymbol{\theta}), \forall K \geqslant K_1, \forall \boldsymbol{\theta}$. Since $y(K,\boldsymbol{\theta}) - C(p^*(K,\boldsymbol{\theta}),K,\boldsymbol{\theta}) \geqslant K_1$, then for all $K \geqslant K_1$ and $\boldsymbol{\theta}$,$\boldsymbol{\theta}$':

$$p\left(y(K,\boldsymbol{\theta}) - C(p^*(K,\boldsymbol{\theta}),K,\boldsymbol{\theta}),\boldsymbol{\theta}'\right) \leqslant p^*\left(y(K,\boldsymbol{\theta}) - C(p^*(K,\boldsymbol{\theta}),K,\boldsymbol{\theta}),\boldsymbol{\theta}'\right)$$

Therefore, $Z(p^*(K,\boldsymbol{\theta}),p^*(K,\boldsymbol{\theta}),K,\boldsymbol{\theta}) = 0 \geqslant Z(p^*(K,\boldsymbol{\theta}),p(K,\boldsymbol{\theta}),K,\boldsymbol{\theta})$, which implies that $Tp(K,\boldsymbol{\theta}) \leqslant p^*(K,\boldsymbol{\theta})$. T is K_0-monotone. □

Lemma A8. *Under the conditions of lemma A4 and assumption 2, T is pseudo-concave.*

Proof. We want for $t \in (0,1)$ that $T[tp](K,\boldsymbol{\theta}) > tT[p](K,\boldsymbol{\theta})$ for $K > 0$. Since Z is strictly decreasing in p, it is equivalent to show that

$$0 = Z(T[tp](K,\boldsymbol{\theta}),tp,K,\boldsymbol{\theta}) < Z(tT[p](K,\boldsymbol{\theta}),tp,K,\boldsymbol{\theta}).$$

$$\begin{aligned}
&Z(tT[p](K,\boldsymbol{\theta}),tp,K,\boldsymbol{\theta}) \\
&= \frac{1}{tT[p](K,\boldsymbol{\theta})} - \beta\mathbb{E}\left[\frac{r\left(y(K,\boldsymbol{\theta}) - C(tT[p](K,\boldsymbol{\theta}),K,\boldsymbol{\theta}),\boldsymbol{\theta}'\right)}{tp\left(y(K,\boldsymbol{\theta}) - C(tT[p](K,\boldsymbol{\theta}),K,\boldsymbol{\theta}),\boldsymbol{\theta}'\right)}\right] \\
&= \frac{1}{t}\left\{\frac{1}{T[p](K,\boldsymbol{\theta})} - \beta\mathbb{E}\left[\frac{r\left(y(K,\boldsymbol{\theta}) - C(tT[p](K,\boldsymbol{\theta}),K,\boldsymbol{\theta}),\boldsymbol{\theta}'\right)}{p\left(y(K,\boldsymbol{\theta}) - C(tT[p](K,\boldsymbol{\theta}),K,\boldsymbol{\theta}),\boldsymbol{\theta}'\right)}\right]\right\},
\end{aligned}$$

since C is strictly increasing in p, $C\left(tT\left[p\right]\left(K,\boldsymbol{\theta}\right),K,\boldsymbol{\theta}\right) < C\left(T\left[p\right]\left(K,\boldsymbol{\theta}\right),K,\boldsymbol{\theta}\right)$. Since $\frac{r}{p}$ is strictly decreasing in K, then

$$\begin{aligned} Z\left(tAv, tv, K, \boldsymbol{\theta}\right) &= \frac{1}{t}\left\{\frac{1}{T\left[p\right]\left(K,\boldsymbol{\theta}\right)} - \beta\mathbb{E}\left[\frac{r\left(y\left(K,\boldsymbol{\theta}\right) - C\left(tT\left[p\right]\left(K,\boldsymbol{\theta}\right),K,\boldsymbol{\theta}\right),\boldsymbol{\theta}'\right)}{p\left(y\left(K,\boldsymbol{\theta}\right) - C\left(tT\left[p\right]\left(K,\boldsymbol{\theta}\right),K,\boldsymbol{\theta}\right),\boldsymbol{\theta}'\right)}\right]\right\} \\ &> \frac{1}{T\left[p\right]\left(K,\boldsymbol{\theta}\right)} - \beta\mathbb{E}\left[\frac{r\left(y\left(K,\boldsymbol{\theta}\right) - C\left(T\left[p\right]\left(K,\boldsymbol{\theta}\right),K,\boldsymbol{\theta}\right),\boldsymbol{\theta}'\right)}{p\left(y\left(K,\boldsymbol{\theta}\right) - C\left(T\left[p\right]\left(K,\boldsymbol{\theta}\right),K,\boldsymbol{\theta}\right),\boldsymbol{\theta}'\right)}\right] = 0, \end{aligned}$$

which shows that $T\left[tp\right]\left(K,\boldsymbol{\theta}\right) > tT\left[p\right]\left(K,\boldsymbol{\theta}\right)$ for $K > 0$. Therefore, T is a pseudo-concave mapping. □

Proposition A3. *Under the conditions of proposition A1, assumptions 1-2 and ω close enough to 1 such that $y\left(K,\boldsymbol{\theta}\right) - \frac{L(K,\boldsymbol{\theta})^{1+\nu}}{1+\nu}$ is nondecreasing in K and R is nonincreasing in K, there is a unique strictly positive equilibrium $p \in \mathscr{P}$.*

Proof. T being K_0-monotone, pseudo-concave has at most one strictly positive fixed point. Take two fixed points p_1^* and p_2^*. Suppose for some $K > 0, \boldsymbol{\theta} \in \Theta^2$, $p_1^*\left(K,\boldsymbol{\theta}\right) < p_2^*\left(K,\boldsymbol{\theta}\right)$. Pick the K_0 from the K_0-monotonicity and choose $t \in (0,1)$ such that $p_1^*\left(K,\boldsymbol{\theta}\right) \geqslant tp_2^*\left(K,\boldsymbol{\theta}\right)$, i.e., choose $t = \inf_{K \geqslant K_0} \frac{p_1^*(K,\boldsymbol{\theta})}{p_2^*(K,\boldsymbol{\theta})}$ which is finite, strictly positive (recall that the p's are increasing in K, strictly positive and bounded) and strictly less than 1 by assumption. Then, by K_0-monotonicity, for all $K \geqslant K_0$,

$$\begin{aligned} p_1^*\left(K,\boldsymbol{\theta}\right) &\geqslant T\left[tp_2^*\right]\left(K,\boldsymbol{\theta}\right) \\ &> tT\left[p_2^*\right]\left(K,\boldsymbol{\theta}\right) \\ &> tp_2^*\left(K,\boldsymbol{\theta}\right) \end{aligned}$$

which contradicts the fact that t was the infimum. Therefore, the equilibrium is unique. □

F.5 Policy

Proposition 5. *The competitive equilibrium with incomplete information is inefficient, but the constrained efficient allocation can be implemented with a lump-sum tax on the household, an input subsidy s_{kl} and a profit subsidy s_π to intermediate goods producers such that $1 - s_{kl} = \frac{\sigma-1}{\sigma}$ and $1 + s_\pi = \frac{\sigma}{\sigma-1}$.*

Proof. We define the constrained planner problem as selecting a schedule $z\left(v_j\right)$ of probabilities to use high capacity as a function of an agent's private signal v_j and levels of production Y_h and Y_l for capacity levels. Define the planner's Bellman equation

$$V_{SP}(K,\theta_{-1}) = \max_{0\leqslant z(\cdot)\leqslant 1} \mathbb{E}_\theta \left[\max_{K',L,K_i,L_i} U\left(Y - m(\theta,z) f - K' + (1-\delta)K, L\right) \right.$$
$$\left. + \beta V_{SP}\left(K',\theta\right) \mid \theta_{-1} \right]$$

subject to

$$\begin{aligned}
Y &= \left(\int_0^1 m(\theta,z) Y_h^{\frac{\sigma-1}{\sigma}} + (1-m(\theta,z)) Y_l^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \\
Y_i &= A_i(\theta) K_i^\alpha L_i^{1-\alpha}, i \in \{h,l\} \\
K &= m(\theta,z) K_h + (1-m(\theta,z)) K_l \\
L &= m(\theta,z) L_h + (1-m(\theta,z)) L_l \\
m(\theta,z) &= \int \sqrt{\gamma_v}\phi\left(\sqrt{\gamma_v}(v-\theta)\right) z(v) dv.
\end{aligned}$$

The first-order conditions with respect to K_i and L_i tell us that the marginal products are equalized across firms,

$$\begin{aligned}
\alpha Y_h^{\frac{\sigma-1}{\sigma}-1} Y^{\frac{1}{\sigma}} \frac{Y_h}{K_h} &= \alpha Y_l^{\frac{\sigma-1}{\sigma}-1} Y^{\frac{1}{\sigma}} \frac{Y_l}{K_l} \\
(1-\alpha) Y_h^{\frac{\sigma-1}{\sigma}-1} Y^{\frac{1}{\sigma}} \frac{Y_h}{L_h} &= (1-\alpha) Y_l^{\frac{\sigma-1}{\sigma}-1} Y^{\frac{1}{\sigma}} \frac{Y_l}{L_l},
\end{aligned}$$

and we have equality between the marginal product of labor and the marginal rate of substitution,

$$(1-\alpha) Y_i^{\frac{\sigma-1}{\sigma}-1} Y^{\frac{1}{\sigma}} \frac{Y_i}{L_i} = \frac{U_L(C,L)}{U_C(C,L)} = L^\nu.$$

Solving this system of equation, we obtain the following efficient output level and labor,

$$Y_{SP}(K,\theta,m) = (1-\alpha)^{\frac{1-\alpha}{\alpha+\nu}} \left(\overline{A}(\theta,m) K^\alpha\right)^{\frac{1+\nu}{\alpha+\nu}} \text{ and } L_{SP}(K,\theta,m) = (1-\alpha)^{\frac{1}{\alpha+\nu}} \left(\overline{A}(\theta,m) K^\alpha\right)^{\frac{1}{\alpha+\nu}}. \tag{32}$$

The first order condition on $z(v)$ is

$$\mathbb{E}_\theta \left[\sqrt{\gamma_v}\phi\left(\sqrt{\gamma_v}(\theta - v)\right) U_C(C,L) \left(\frac{\overline{A}_m(\theta,m)}{\overline{A}(\theta,m)} Y_{SP} - f \right) \mid \theta_{-1} \right] \gtreqless 0,$$

with corresponding complementary slackness conditions. Substituting in the values of $\overline{A}$ and Y_{SP}, we get

$$\mathbb{E}_\theta \left[\sqrt{\gamma_v}\phi\left(\sqrt{\gamma_v}(\theta - v)\right) U_C(C,L) \left(\frac{1}{\sigma-1} \frac{\omega^{\sigma-1}-1}{\Omega(m)^{\sigma-1}} (1-\alpha)^{\frac{1-\alpha}{\alpha+\nu}} \left(\overline{A}(\theta,m) K^\alpha\right)^{\frac{1+\nu}{\alpha+\nu}} - f \right) \mid \theta_{-1} \right] \gtreqless 0.$$

It is now useful to notice that, since $\theta = \rho\theta_{-1} + \sqrt{\gamma_\theta}\varepsilon_\theta$ and $v_j = \theta + \sqrt{\gamma_v}\varepsilon_{vj}$, with $(\varepsilon_\theta, \varepsilon_{vj})$ unit normals, then for any arbitrary function $f(K, \theta, m)$, the following equality holds

$$\begin{aligned}\mathbb{E}_\theta\left[f(K,\theta,m) \mid \theta_{-1}, v_j\right] &= \int f(K,\theta,m)\sqrt{\gamma_\theta}\phi\left(\sqrt{\gamma_\theta}(\theta-\rho\theta_{-1})\right)\sqrt{\gamma_v}\phi\left(\sqrt{\gamma_v}(v-\theta)\right)d\theta/\pi(v \mid \theta_{-1}) \\ &= \mathbb{E}_\theta\left[f(K,\theta,m)\sqrt{\gamma_v}\phi\left(\sqrt{\gamma_v}(v-\theta)\right) \mid \theta_{-1}\right]/\pi(v \mid \theta_{-1}).\end{aligned}$$

Thus, going back to the planner's problem, a firm with signal v chooses high capacity with positive probability if and only if

$$\mathbb{E}_\theta\left[U_C(C,L)\left(\frac{1}{\sigma-1}\frac{\omega^{\sigma-1}-1}{\Omega(m)^{\sigma-1}}(1-\alpha)^{\frac{1-\alpha}{\alpha+\nu}}\left(\overline{A}(\theta,m)K^\alpha\right)^{\frac{1+\nu}{\alpha+\nu}} - f\right) \mid \theta_{-1}, v_j\right] \geqslant 0.$$

This problem is familiar and we recognize a condition similar to the one that characterizes the solution of the global game. Using the same arguments as before, we know that the stochastic discount factor U_C drops from the equation when $\gamma_v \to \infty$, which simplifies the problem to

$$\mathbb{E}_\theta\left[\frac{1}{\sigma-1}\frac{\omega^{\sigma-1}-1}{\Omega(m)^{\sigma-1}}(1-\alpha)^{\frac{1-\alpha}{\alpha+\nu}}\left(\overline{A}(\theta,m)K^\alpha\right)^{\frac{1+\nu}{\alpha+\nu}} - f \mid \theta_{-1}, v\right] \geqslant 0.$$

Then, under the same hypothesis as proposition A1, i.e., that $\frac{\sqrt{\gamma_v}}{\gamma_\theta} > \frac{1}{\sqrt{2\pi}}\frac{\omega^{\sigma-1}-1}{\sigma-1}$, we know that the only solution to the above equation is a cutoff $\hat{v}_{SP}(K, \theta_{-1})$ such that $z(v) = 1$ for $v > \hat{v}_{SP}(K, \theta_{-1})$ and $z(v) = 0$ for $v < \hat{v}_{SP}(K, \theta_{-1})$. The cutoff is such that

$$\mathbb{E}_\theta\left[\frac{1}{\sigma-1}\frac{\omega^{\sigma-1}-1}{\Omega(m)^{\sigma-1}}(1-\alpha)^{\frac{1-\alpha}{\alpha+\nu}}\left(\overline{A}\left(\theta, \Phi\left(\sqrt{\gamma_v}(\theta-\hat{v}_{SP})\right)\right)K^\alpha\right)^{\frac{1+\nu}{\alpha+\nu}} - f \mid \theta_{-1}, \hat{v}_{SP}\right] = 0. \tag{33}$$

Comparing the two conditions (32) and (33) to that of the competitive economy, we see that the conditions coincide either with the input subsidy $1 - s_{kl} = \frac{\sigma-1}{\sigma}$ so as to offset the markup and the profit subsidy $1 + s_\pi = \frac{\sigma}{\sigma-1}$ to induce the right entry; or more simply, just using a sales subsidy $1 + s_y = \frac{\sigma}{\sigma-1}$. Under the conditions of proposition A3, we know that these two first order conditions uniquely determine the equilibrium. Therefore, the two economies coincide under this optimal sales subsidy and the economy without subsidy is inefficient. □

Proposition 6. *Under GHH preferences, for γ_v large, an unforeseen one-time increase in government spending financed by lump-sum taxes reduces welfare.*

Proof. Consider the case of an unforeseen shock to government spending $G_0 > 0$ that lasts only one period, $G_t = 0$ for $t \geqslant 1$ financed by a lump-sum tax $T_0 = G_0$.[37] Notice that, under our assumption of GHH preferences, our expressions for equilibrium output $Y(K, \theta, m)$ and labor $L(K, \theta, m)$ from proposition 1 remain unaffected by government spending. The only channel by which spending may influence output is through the coordination game by affecting the measure of firms with high capacity m. As shown in lemma A1, as γ_v becomes large, the within-period uncertainty vanishes

[37] Ricardian equivalence obtains in our environment and the actual timing of taxes is irrelevant.

and the stochastic discount factor disappears from the surplus from choosing the high capacity which, in the absence of other subsidies, can be approximated by

$$\Delta\tilde{\Pi}\left(K,\theta_{-1},v_j,m\right)=\mathbb{E}_\theta\left[\frac{1}{\sigma}\frac{\omega^{\sigma-1}-1}{\Omega\left(m\right)^{\sigma-1}}\left[\left(1-\alpha\right)\frac{\sigma-1}{\sigma}\frac{1+s_y}{1-s_{kl}}\right]^{\frac{1-\alpha}{\alpha+\nu}}\overline{A}\left(\theta,m\right)^{\frac{1+\nu}{\alpha+\nu}}K^{\alpha\frac{1+\nu}{\alpha+\nu}}-f\mid\theta_{-1},v_j\right].$$

As a result, when γ_v is large, equilibrium consumption C drops from the equation and the solution $\hat{v}\left(K,\theta_{-1}\right)$ to the global game is independent from government spending G. The equilibrium production Y is thus unaffected.

Consider now the equilibrium allocation $\left\{C_t\left(\theta^t\right),K_{t+1}\left(\theta^t\right)\right\}$ in the economy hit by the government spending shock. Because equilibrium production $Y_t\left(\theta^t\right)$ and labor $L_t\left(\theta^t\right)$ are unaffected by government spending, hence prices as well, the same allocation is feasible in an economy without government spending: it satisfies both the household's budget constraint and the aggregate resource constraint with some extra resources left from the unused government consumption. By increasing consumption in period 0 by G_0 exactly, the household can choose an allocation that remains feasible and strictly increases its welfare. As a conclusion, welfare in the economy without spending is strictly greater than in the economy with government spending shock. □