Aggregate Demand and the Dynamics of Unemployment

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Abstract

We introduce an aggregate demand externality into the Mortensen-Pissarides model of equilibrium unemployment. Because firms care about the demand for their products, an increase in unemployment lowers the incentives to post vacancies which further increases unemployment. This positive feedback creates a coordination problem among firms and leads to multiple equilibria. We show, however, that the multiplicity disappears when enough heterogeneity is introduced in the model. In this case, the unique equilibrium still exhibits interesting dynamic properties. In particular, the importance of the aggregate demand channel grows with the size and duration of shocks, and multiple stationary points in the dynamics of unemployment can exist. We calibrate the model to the U.S. economy and show that the mechanism generates additional volatility and persistence in labor market variables, in line with the data. In particular, the model can generate deep, long-lasting unemployment crises.

JEL Classifications: E24, D83

1 Introduction

The slow recovery that followed the Great Recession of 2007-2009 has revived interest in the long-held view in macroeconomics that episodes of high unemployment can persist for extended periods of time because of depressed aggregate demand. The mechanism seems intuitive: when firms expect lower demand for their products, they refrain from hiring and unemployment increases. In turn, as unemployment rises, aggregate income and spending decline, effectively confirming the low aggregate demand. In this paper, we propose a theory of unemployment and aggregate demand to investigate this mechanism. We find that it has a limited impact on the economy in normal timesbut that, because of it, large shocks can push the economy into deep, long-lasting recessions.

The theory augments the benchmark search and matching framework of Mortensen and Pissarides (MP) (Pissarides, 2000) with monopolistic competition and a CES demand system (Dixit and Stiglitz, 1977). As a result, the level of aggregate demand matters when firms make their hiring decisions. In particular, if they expect unemployment to be high, they understand that aggregate spending will be low and that the demand for their products will also be low. Under such expectations, firms post fewer vacancies which leads to high unemployment. This positive feedback loop between employment and aggregate demand amplifies the impact of shocks on the economy. As unemployment evolves slowly over time, and that future unemployment affects future demand, the mechanism also generates propagation.

The aggregate demand channel creates a coordination problem that naturally leads to multiple equilibria. Intuitively, a firm might not want to post a vacancy when other firms are also not hiring since unemployment would then be high and aggregate demand low. The same firm might, however, be willing to post a vacancy when other firms are doing the same. In this case, unemployment would be low and aggregate demand high, providing enough revenue to cover the vacancy cost.

This multiplicity of equilibria is problematic to evaluate the quantitative properties of the model and the impact of policies. We show, however, that the multiplicity is sensitive to the assumptions that firms are homogeneous. Indeed, we demonstrate that uniqueness obtains when firms are heterogeneous in vacancy posting costs and that the distribution of these costs is sufficiently dispersed. Despite the lack of multiplicity, the model retains interesting features due to the complementarities in aggregate demand. We thus argue that it is possible to study models with strong complementarities without leaving the realm of unique equilibria.

The unique equilibrium of this economy features rich dynamic properties. First, owing to the need for firms to forecast aggregate demand, the level of unemployment is a state variable, unlike in the MP model. Second, for a given level of productivity, multiple steady states and attractors may arise despite equilibrium uniqueness. As a result, the dynamics of unemployment may sometimes become unstable and the economy may go through large unemployment crisis as it momentarily diverges to some highly depressed attractor. In particular, when the demand complementarity is strong enough, multiple steady states in the dynamics of the unemployment rate can arise.

We calibrate the model to the U.S. economy and compare its dynamic properties to the the data. We find that the aggregate demand channel improves on the MP model by generating volatility of unemployment and vacancy levels similar to the data. Additionally, the model generates strong propagation of productivity shocks more in line with the data. Finally, we show that a Depression-era style of recession can arise in the model after sufficiently large shocks.

The paper contributes to the literature on dynamic models of coordination (Chamley, 1999; Angeletos et al., 2007). It relates to a previous paper of ours, Schaal and Taschereau-Dumouchel (2015), in which we study coordination failures in a real business cycle model with an aggregate demand externality and a nonconvex technology choice. Using a global game approach to discipline equilibrium selection, we show that multiplicity in steady states can arise, opening up the way to coordination traps. In this paper, we consider the same source of complementarity in demand, but study instead its implications for labor market fluctuations. We also use a different approach for equilibrium selection by showing that a simple form of heterogeneity can guarantee uniqueness. Because of the possibility of multiple steady states in our model, our paper is also related to Sterk (2015), who estimates a reduced form non-linear dynamic model on labor market flow data. His findings suggest the existence of multiple steady states in the dynamics of unemployment.

This paper is also related to a literature that considers the role of externalities in search models. Diamond (1982) shows that a search model with a thick-market externality can generate multiple steady-state equilibria. Diamond and Fudenberg (1989) study the dynamic properties of that model and find that multiple equilibrium paths may exist. Howitt and McAfee (1992) find that animal spirits can generate unemployment fluctuations in an economy with a thick-market externality. Mortensen (1999) considers a search model with increasing returns to scale in production and shows that multiple equilibria arise, some with limit cycles. Sniekers (2014) shows that similar limit cycles can generate additional volatility and persistence without relying on exogenous shocks. More recent contributions include Kaplan and Menzio (2014) and Eeckhout et al. (2015). Kaplan and Menzio (2014) explore how shopping externalities can give rise to multiple equilibrium paths. Eeckhout et al. (2015) study how on-the-job search effort combines with a thick-market externality to create self-fulfilling unemployment fluctuations.

This paper contributes to the literature that has sought to solve the unemployment volatility puzzle (Shimer, 2005). To increase the volatility of unemployment and vacancies in the MP model, several authors have modified the wage setting process to make it less responsive to productivity shocks. Hall and Milgrom (2008) departs from traditional Nash-bargaining by assuming that agents can extend the bargaining process instead of terminating it. As a result, wages depend less on the outside options, which increases the volatility in unemployment and vacancies. Gertler and Trigari (2009) show that a search model with staggered multi-period wage contracting can account for the cyclical behavior of various labor market indicators. In contrast to these two papers, we rely on an aggregate demand channel to generate additional volatility. Pissarides (2009) is critical of the wage rigidity approach. He shows that only the wage of new matches matter for vacancy posting and then documents that these wages are strongly pro-cyclical in the data, as they are in our calibrated economy. Hagedorn and Manovskii (2008) show that a calibration of the MP model with a high labor supply elasticity can generate unemployment and vacancies that fluctuate enough to match the data. Their parametrization, however, implies a large Frisch elasticity of labor supply (Hall and Milgrom, 2008). In contrast, our parametrization implies a smaller elasticity and non-linear dynamics.

The next section presents the model. Section 3 discusses the possibility of multiple equilibria and provides conditions under which multiplicity or uniqueness may obtain. We calibrate the model in Section 4 and study the impact of the aggregate demand channel on its dynamic properties. The last section concludes. All proofs are in the Appendix.

2 Model

We begin by introducing the model, which is a simple extension of the standard Mortensen-Pissarides model (MP) with monopolistic competition and heterogeneity in vacancy costs.

2.1 Environment

Time is discrete and goes on forever. There are two types of goods: a final good used for consumptions and a continuum of intermediate goods used in the production of the final good.

There is a unit mass of workers and an endogenous measure of firms. Workers and firms are risk-neutral and discount future consumption of the final good at the same rate $0 < \beta < 1$. A fraction s of the workers are self-employed: they can produce one variety of intermediate goods without having to combine their labor with the technology of a firm. The remaining $1-s$ workers must be employed by a firm to produce. Unemployed workers enjoy leisure valued as b units of the final good. We denote by n the mass of such workers who produce in a given period and $u = 1 - s - n$ the mass of unemployed workers.[1]

Each employed worker $j \in [0, 1-u]$ produces $Y_j = Ae^z$ units of variety j of intermediate goods. Aggregate productivity z follows an AR(1) process

$$z' = \rho z + \varepsilon_z, \tag{1}$$

with $\varepsilon_z \sim$ iid $\mathcal{N}\left(0, \left(1-\rho^2\right)\sigma_z^2\right)$ and $A > 0$ is a constant.

[1]The presence of self-employed workers is not essential for the mechanism itself but prevents the unemployment rate from reaching the value of 1 at which point it is impossible to rule out multiple equilibria.

Labor market

Firms and workers looking for a job meet in a frictional labor market characterized by search frictions. If u workers are searching and v vacancies have been posted, a vacancy is filled with probability $q(\theta)$, where $\theta \equiv v/u$ and $q' < 0$, and a worker finds a job with probability $p(\theta) = \theta q(\theta)$, $p' > 0$. Jobs are exogenously destroyed with probability $\delta > 0$.

Timing

The timing of events is as follows

1. The period starts with a new draw of productivity z and u unemployed workers;
2. Production takes place;
3. Firms post vacancies and matches are created. Incumbent jobs are destroyed with probability δ.

With that timing, unemployment u follows the law of motion

$$u' = (1 - p(\theta))\, u + \delta (1 - s - u). \tag{2}$$

Final good producers

The final good is produced by a perfectly competitive, representative firm that combines the continuum of differentiated intermediate goods using the CES production function

$$Y = \left(\int_0^{1-u} Y_j^{\frac{\sigma-1}{\sigma}} dj \right)^{\frac{\sigma}{\sigma-1}}, \tag{3}$$

where $\sigma > 1$ is the elasticity of substitution between varieties, Y is the total output of the final good and Y_j denotes the input of intermediate good j. Profit maximization, taking output price P and input prices P_j as given, yields the usual factor demand curves and the price of the final good,

$$Y_j = \left(\frac{P_j}{P} \right)^{-\sigma} Y \text{ and } P = \left(\int_0^{1-u} P_j^{1-\sigma} dj \right)^{\frac{1}{1-\sigma}}. \tag{4}$$

We normalize $P = 1$ in every period.[2]

[2] Note that the measure of intermediate good varieties fluctuates over time. An alternative is to fix the measure of varieties to one and assume that unemployed workers produce bAe^{z_t} units of a given variety of intermediate good at home with $b < 1$.

Value functions

Because of the aggregate demand externality that results from monopolisitic competition, the unemployment rate is part of the state space of the economy as it governs the level of aggregate demand Y.

After being matched, each worker-firm pair produces a particular variety j of intermediate good. The value of a firm is

$$J(z,u) = P_j Y_j - w + \beta(1-\delta) E\left[J(z',n')\right], \tag{5}$$

where w is the wage. We assume that firms are destroyed after separation. The value of an employed worker is

$$W(z,u) = w + \beta E\left[(1-\delta) W(z',u') + \delta U(z',u')\right], \tag{6}$$

and the value of an unemployed worker is

$$U(z,u) = b + \beta E\left[p(\theta) W(z',u') + (1-p(\theta)) U(z',u')\right]. \tag{7}$$

Wages are set through Nash bargaining between the intermediate good producers and the workers. Denoting γ the bargaining power of workers, the joint surplus is split according to

$$(1-\gamma)\left[W(z,u) - U(z,u)\right] = \gamma J(z,u),$$

yielding the familiar expression

$$w = \gamma P_j Y_j + (1-\gamma) b + \gamma\beta p(\theta) E\left[J(z',u')\right]. \tag{8}$$

Entry problem

Every period, a finite mass M of potential entrants gets to choose whether to enter or not. Posting a vacancy is costly and requires the payment of a random cost κ in units of the final good. The vacancy cost κ is drawn *iid* from the cumulative distribution $F(\kappa)$ with support $[\underline{\kappa}, \overline{\kappa}]$. A potential entrant posts a vacancy if the expected value of a job exceeds its cost,

$$\beta q(\theta) E\left[J(z',u')\right] \geqslant \kappa.$$

The optimal entry decision naturally takes the form of a cutoff rule $\hat{\kappa}(z,u)$ such that firms with costs $\kappa \leqslant \hat{\kappa}(z,u)$ post vacancies. Hence, in equilibrium, the labor market tightness satisfies $\theta = MF(\hat{\kappa})/u$. The marginal entrant $\hat{\kappa}$ must belong to the set $K(z,u)$ such that

$$K(z,u) = \left\{\kappa \in [\underline{\kappa}, \overline{\kappa}] \mid \beta q(\theta) E\left[J\left(z', (1-p(\theta))u + \delta(1-s-u)\right)\right] = \kappa\right\} \tag{9}$$
$$\cup \left\{\overline{\kappa} \text{ if } \beta q\left(\frac{M}{u}\right) E\left[J\left(z', \left(1-p\left(\frac{M}{u}\right)\right)u + \delta(1-s-u)\right)\right] > \overline{\kappa}\right\}$$
$$\cup \left\{\underline{\kappa} \text{ if } \beta q(0) E\left[J\left(z', (1-p(0))u + \delta(1-s-u)\right)\right] < \underline{\kappa}\right\}.$$

Note that these conditions are not mutually exclusive and there can exist multiple solutions to the entry problem, as we show in the next section.

2.2 Equilibrium Definition

We are now ready to define a recursive equilibrium for this economy.

Definition 1. *A recursive equilibrium is a set of value functions for firms $J(z,u)$, for workers $W(z,u)$ and $U(z,u)$, a cutoff rule $\hat{\kappa}(z,u)$ and an equilibrium labor market tightness $\theta(z,u)$ such that*

1. The value functions satisfy equations (5), (6) and (7) under the wage equation (8),
2. The cutoff $\hat{\kappa}$ solves the entry problem, i.e., $\hat{\kappa}(z,u) \in K(z,u)$ as defined in (9),
3. The labor market tightness is such that $\theta(z,u) = MF(\hat{\kappa}(z,u))/u$, and
4. Unemployment follows the law of motion (2).

3 Complementarities, Multiplicity and Non-linear Dynamics

The addition of monopolistic competition into an otherwise standard MP model introduces a new feedback from aggregate demand to job creation. We analyze in this section how this feedback naturally leads to multiplicity of equilibria. While this multiplicity may be interesting per se, the large amount of dynamic sunspot equilibria that arise make the model less amenable to quantitative and policy analysis. We show how including some form of heterogeneity across agents selects a unique equilibrium while retaining the various interesting dynamic implications that complementarities give rise to.

3.1 Complementarity and Multiplicity

Because of monopolistic competition and the CES demand structure in the differentiated goods sector, firms must take into account the level of aggregate demand when posting vacancies. To see this, we substitute the demand curve of the intermediate goods producers (4) into their revenue function to find that the total sales of firm j is $P_{jt}Y_{jt} = Y_t^{\frac{1}{\sigma}} \left(Ae^{z_t}\right)^{1-\frac{1}{\sigma}}$. Using expression (3) and the symmetry across producers, aggregate demand is given by

$$Y = Ae^{z} \left(1-u\right)^{\frac{\sigma}{\sigma-1}}. \tag{10}$$

This aggregate demand externality, as it was termed by Blanchard and Kiyotaki (1987), introduces a Keynesian aggregate demand feedback that reduces firms' incentives to create jobs during downturns. In a recession, aggregate spending falls as unemployment rises. In turn, expectations of future revenue decline, making it less attractive for firms to create jobs. As a result, fewer firms post vacancies and unemployment rises further. Through this mechanism, aggregate demand can amplify and propagate the impact of shocks on the economy.

This positive feedback loop between employment and aggregate demand naturally leads to multiple equilibria. Intuitively, if a firm expected other firms to create jobs, they would anticipate strong aggregate demand for the subsequent periods and would be more likely to post a vacancy. Conversely, if other firms abstained from posting vacancies, the resulting aggregate demand would be expected to be low, weakening the incentives to create a job in the first place.

To understand how this aggregate demand feedback manifests itself in our model, it is useful to examine the entry problem and introduce the following function

$$\Psi\left(z,u,\hat{\kappa}\right) \equiv \underbrace{q\left(\theta\left(\hat{\kappa}\right)\right)}_{(1)}\beta E\left[J\left(z',\left(1-\underbrace{p\left(\theta\left(\hat{\kappa}\right)\right)}_{(2)}u+\delta\left(1-s-u\right)\right)\right)\right]-\underbrace{\hat{\kappa}}_{(3)}, \tag{11}$$

where $\theta = \frac{M}{u}F\left(\hat{\kappa}\right)$. The function Ψ captures the incentives to post a vacancy for the marginal entrant $\hat{\kappa}$: a value of Ψ greater than 0 implies that posting vacancies remains attractive for the marginal firm and that more firms should enter. Solutions to the entry problem are such that the marginal entrant with cost $\hat{\kappa}\left(z,u\right)$ satisfies i) $\Psi\left(z,u,\hat{\kappa}\right) = 0$ if the solution is interior, ii) $\Psi\left(z,u,\underline{\kappa}\right) < 0$ if there is no entry, and iii)$\Psi\left(z,u,\overline{\kappa}\right) > 0$ if there is full entry.[3] The terms (1), (2) and (3) highlight the different motives at work behind job creation. Term (1) is the usual crowding out effect that has been extensively studied in the literature. As more firms post vacancies, they crowd out each other, diminishing the probability that each individual firm gets matched and reducing incentives to create jobs. Term (2) captures the aggregate demand channel. In the presence of monopolistic competition, the unemployment rate becomes a state variable and the value of a firm J depends negatively on the unemployment rate. As a result, an increase in vacancy posting, through an increase in the marginal $\hat{\kappa}$, leads to a reduction in future unemployment that increases the value of the firm. Term (3) results from the heterogeneity in vacancy costs: a rise in the number of entrants requires to move up in the distribution of costs, making it less attractive to

[3] In practice, we choose the mass M of potential entrants large enough that case (iii) does not happen.

create jobs. Terms (1) and (3) capture substitutabilities: they correspond to stabilizing forces that decrease the value of posting a vacancy when $\hat{\kappa}$ goes up. Term (2) embodies a complementarity: it is a destabilizing force that encourages firms to post more vacancies when others do. A race between these forces determines whether or not multiple equilibria can exist.

Figure 1 illustrates how these forces combine to deliver uniqueness or multiplicity. Panel (a) displays the function Ψ in two different cases. The dashed black line corresponds to the case in which complementarities are turned off ($\sigma = \infty$), isolating the effect of terms (1) and (3) alone. The continuous blue curve displays a case in which the aggregate demand channel is active ($\sigma \ll \infty$), capturing the additional impact of term (2). Panel (b) displays the marginal density function of the vacancy costs in an effort to highlight how the shape of the function Ψ depends on the measure of entrants. For most values of $\hat{\kappa}$, Ψ is decreasing almost linearly, reflecting the contribution of term (3). However, when the measure of entrants start to increase, as in the case indicated with the two white circles, terms (1) and (2) kick in. In the absence of the aggregate demand channel, the crowding out effect dominates and pushes the value of posting a vacancy further down, leading to a unique solution to the entry problem. In the presence of the aggregate demand channel (2), when complementarities are large enough to dominate the substitutability forces, Ψ may curve back up, opening up the possibility of multiple solutions to the entry problem.

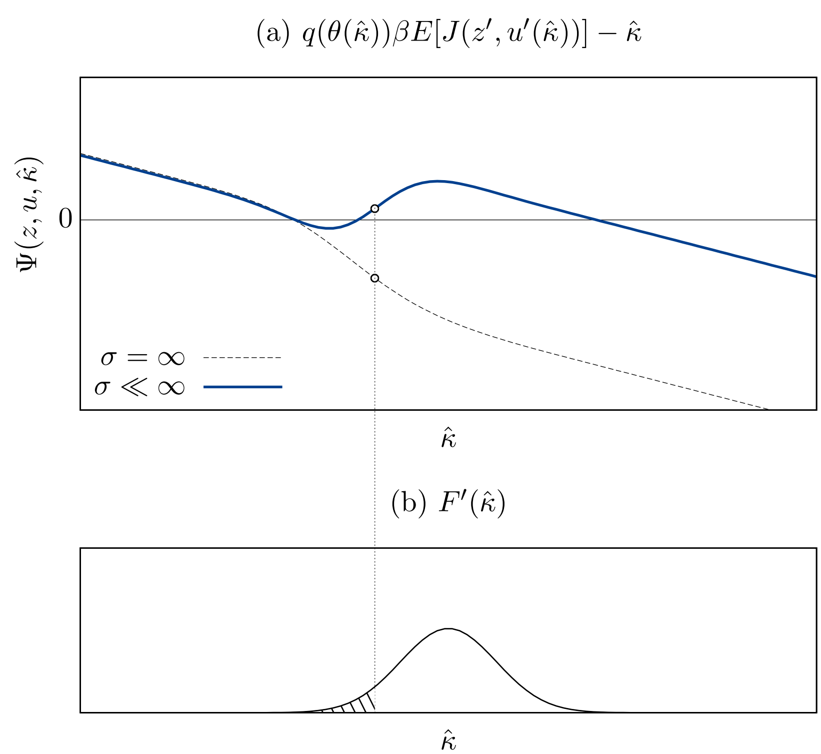


Figure 1: Role of the complementarity for multiplicity

While the above discussion relied on qualitative properties of the model, it is particularly easy to show an example of this multiplicity under some simplifying assumptions.

Example. Consider a deterministic economy in which i) the fundamental z is fixed, ii) firms have all the bargaining power $\gamma = 0$, iii) the matching function is such that $p(\theta) = \min\{1, \theta\}$ and $q(\theta) = \min\left\{\theta^{-1}, 1\right\}$, iv) there are no self-employed workers $s = 0$ and v) vacancy costs are homogeneous and equal to κ. We will construct two equilibria: a *normal* stationary equilibrium and a *depression* equilibrium that starts from the *normal* steady state and in which firms post no vacancies thereafter. In a steady state with unemployment $\overline{u}$, the value of a firm is

$$J\left(u\right) = \left(Ae^{z}\left(1-\overline{u}\right)^{\frac{1}{\sigma-1}} - b\right),$$

while the unemployment rate must satisfy $\overline{u} = \delta/\left(\delta + p\left(\theta\right)\right)$. Combining these values for an interior condition to the entry problem, we find that the steady-state labor market tightness θ must satisfy

$$\frac{\kappa}{q\left(\theta\right)} = \frac{1}{1-\beta\left(1-\delta\right)}\left(Ae^{z}\left(\frac{p\left(\theta\right)}{\delta + p\left(\theta\right)}\right)^{\frac{1}{\sigma-1}} - b\right).$$

Looking for a stationary equilibrium tightness $\theta > 1$ we find

$$\bar{\theta} = \frac{1}{\kappa}\frac{1}{1-\beta\left(1-\delta\right)}\left(Ae^{z}\left(\frac{1}{\delta+1}\right)^{\frac{1}{\sigma-1}} - b\right)$$

which exists if z is high enough. Note that the steady-state unemployment rate is $\bar{u} = \delta/\left(\delta+1\right)$. Now let us consider the depression equilibrium that starts from $n_0 = \overline{n} = 1 - \bar{u} = 1/\left(1+\delta\right)$ and such that $v_t = 0$ for all $t \geqslant 0$. Then, $\theta_t = 0$ for all $t > 0$ and $n_t = \left(1-\delta\right)^t n_0$. Firms do not want to deviate from this equilibrium if $\beta J_{t+1}\left(z, n_t\right) < \kappa$ for all t where

$$J_t\left(z, n_t\right) = \sum_{s=0}^{\infty}\left(\beta\left(1-\delta\right)\right)^s\left(Ae^{z}\left(n_{t+s}\right)^{\frac{1}{\sigma-1}} - b\right).$$

Notice that J_t is decreasing in n_t so we only need to verify the condition for $t = 0$. Simplifying the summation, this condition becomes

$$\beta\left[\frac{\left(\left(1-\delta\right)\overline{n}\right)^{\frac{1}{\sigma-1}}Ae^{z}}{1-\beta\left(1-\delta\right)^{\frac{\sigma}{\sigma-1}}} - \frac{b}{1-\beta\left(1-\delta\right)}\right] < \kappa.$$

Notice that with $\sigma = \infty$, the condition is not satisfied for $Ae^{z} - b > \kappa$, which would be the standard requirement to have a non-zero output equilibrium in a standard MP model. When $\sigma < \infty$, however, the condition will be satisfied either for i) σ sufficiently close to 1, ii) β sufficiently close to 1, or iii) δ sufficiently close to 1. The intuition for these limits is straightforward. When σ is close to

1, the degree of complementarities is large, ensuring that the aggregate demand channel dominates all other forces. When β is close to 1, agents are extremely patient and put more weight on future periods when aggregate demand falls apart. Finally, when δ is close to 1, today's vacancy posting has a large impact on tomorrow's aggregate demand, which can easily collapse if not enough jobs are created. As a result, both equilibria can be sustained.

This example, while extreme, illustrates the multiplicity generated by the aggregate demand channel: expectations of high vacancy posting sustains high vacancy posting.

Theoretical Results

In our opening discussion of multiplicity, we mostly focused our attention on the potential multiplicity of solutions to the entry problem. Yet, as the above example suggests, multiplicity can also arise in a dynamic framework in the form of multiple *self-sustaining* value functions, even when the entry problem admits a unique solution. Uniqueness thus requires ruling out both sources of multiplicity. We conclude this section by providing sufficient conditions that guarantee that i) the entry problem admits a unique solution and that ii) the value function is unique. Heterogeneity in vacancy costs is momentarily shut down, and we defer the discussion of its role to the next section.

Proposition 1. *Under some regularity assumptions stated in the appendix and if there is no heterogeneity in vacancy costs, we have the following:*

1. *In the case without aggregate demand externality, $\sigma = \infty$, the equilibrium exists and is unique.*
2. *In the case with $\sigma < \infty$, if there exists $0 < \eta < 1 - (1-\delta)^2$ such that for all (u, θ),*

$$\beta \overline{J}_u u p(\theta) \varepsilon_{p,\theta} \leqslant \eta \frac{\kappa}{q(\theta)} \varepsilon_{q,\theta}, \tag{12}$$

where $\overline{J}_u = (1-\beta)^{-1} \left[(1-\gamma) A e^{\overline{z}} \frac{s^{-\frac{\sigma}{\sigma-1}}}{\sigma-1} + \beta \frac{\eta}{1-\eta} \frac{1-\delta}{\delta} \frac{\gamma}{1-s} \overline{J} \right]$ is an upper bound on the derivative of J with respect to u and $\overline{J}$ an upper bound on J provided in the appendix, then there exists a unique equilibrium if for all (u, θ),

$$\frac{\beta}{1-\eta} \left| 1 - \delta - \gamma p(\theta) \left(1 + \frac{\varepsilon_{p,\theta}}{\varepsilon_{q,\theta}} \right) \right| < 1, \tag{13}$$

where $\varepsilon_{p,\theta} = \frac{dp}{d\theta} \frac{\theta}{p(\theta)}$ and $\varepsilon_{q,\theta} = -\frac{dq}{d\theta} \frac{\theta}{q(\theta)}$ are the elasticities of the matching probabilities.

Proposition 1 begins by stating the well-known result from Mortensen and Nagypal (2007) that without the aggregate demand channel, when $\sigma = \infty$, the model is a standard MP model that admits a unique equilibrium. That case corresponds to the dashed black line on Figure 1. In the absence of complementarities, all the forces in the model contribute to substitutability between agents: the solution to the entry problem is unique and the mapping for the value function is a contraction.

Uniqueness in the case with complementarities requires much stronger conditions on parameters. Condition (12) illustrates the race between complementarities and substitutabilities. In particular, the left hand side captures the strength of the aggregate demand channel, identified as term (2) in equation (11). The right hand side reflects the counteracting crowding out effect, labeled as term (1) in the same equation. Expressed in the form of a race between elasticities with respect to a change in θ, this condition simply states that uniqueness requires the aggregate demand channel to be dominated by the crowding out effects — in fact, not to exceed a fraction $\eta < 1$ of that force in terms of strength. While $\eta = 1$ would be sufficient to guarantee uniqueness in the entry problem, the requirement that $\eta < 1-(1-\delta)^2$, a rather small number for reasonable calibrations, reflects the fact that, in a dynamic setup, future complementarities amplify the current complementarities. In other words, the anticipation of future complementarities in demand gets built in the value function and magnifies their strength, calling for more restrictive conditions for uniqueness. Specifically, condition (12) ensures that the slope of the value function in terms of unemployment does not explode. Finally, equation (13) is the condition that guarantees that the mapping for the value function is a contraction, i.e., that the value function is unique. This conditions highlights that uniqueness is compromised when complementarities are strong (η large). With η close to 1, this condition is unlikely to be satisfied and the mapping for the value functions may admit distinct fixed points.

Note that these theoretical results only provide *sufficient* conditions to obtain a contraction. Not having a contraction does not mean that multiplicity necessarily arises. These results convey, however, the general idea that uniqueness is harder to achieve when complementarities are strong.

3.2 Heterogeneity and Non-linear Dynamics

While it may lead to rich dynamics under sunspots, equilibrium indeterminacy raises a number of issues related to selection that make models with multiple equilibria less amenable to quantitative and welfare analysis than models with a unique equilibrium. An objective of this paper is to show that one can develop models with powerful complementarities without giving up the convenience of equilibrium uniqueness. We show, in particular, that equilibrium multiplicity is fragile and sensitive to the introduction of heterogeneity. We argue, in this section, that a sufficient amount of heterogeneity can lead to a unique equilibrium, while preserving interesting dynamic properties that complementarities bring about.

Returning to our previous discussion on the source of multiplicity in the entry problem, Figure 2 illustrates how heterogeneity affects the entry decision. Panel (a) displays the function Ψ for three different levels of the dispersion σ_κ in vacancy costs under the assumption that the complementarities are large enough to dominate the crowding out effects. The corresponding distributions of costs are shown below in panel (b). As the figure illustrates, a more concentrated distribution of vacancy costs (dotted black line) accelerates the rate of entry when firms start to enter. As a result, the feedback from aggregate demand is magnified, providing support for multiple solutions to exist. More specifically, Ψ is more likely to cross the x-axis several times when the degree of heterogeneity is low. When the distribution sufficiently widens (σ_κ high, continuous blue curve), this feedback weakens up to a point where Ψ flattens and becomes globally decreasing. At this point, the entry problem admits a unique solution, but that solution retains the non-linear shape of Ψ, opening the way for amplification, propagation and non-linear dynamics.

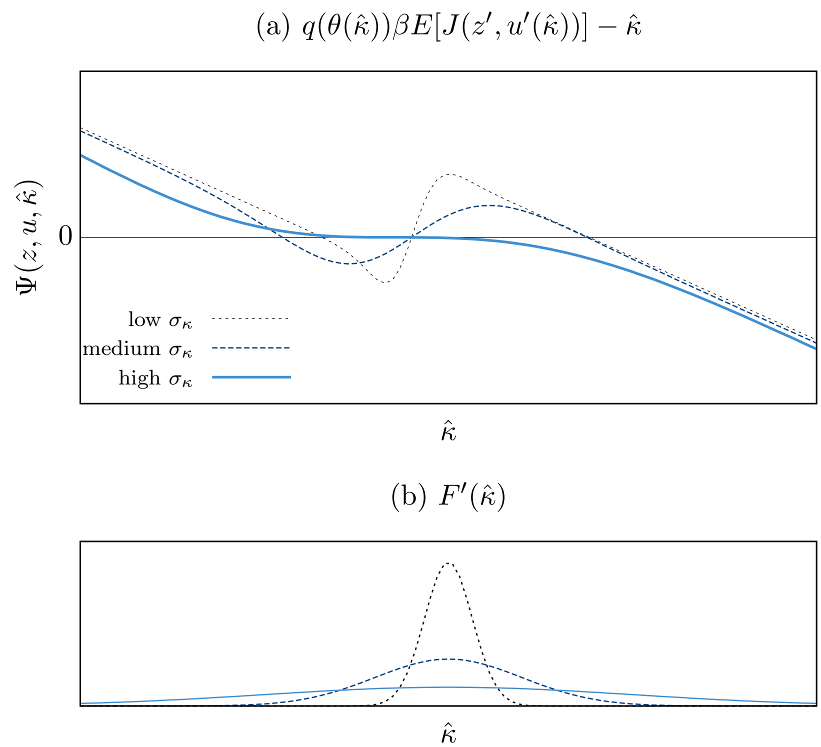


Figure 2: Role of heterogeneity for uniqueness

Theoretical results

We confirm the above intuition that multiplicity vanishes as heterogeneity rises with the following proposition.

Proposition 2. *Under some regularity assumptions stated in the appendix and if there exists*

$0 < \eta < 1 - (1-\delta)^2$ *such that for all* (u, θ),

$$\beta \overline{J}_u u p(\theta) \varepsilon_{p,\theta} \leqslant \eta \frac{\kappa(\theta, u)}{q(\theta)} (\varepsilon_{q,\theta} + \varepsilon_{\kappa,\theta}), \tag{14}$$

where $\varepsilon_{\kappa,\theta} = \frac{d\kappa}{d\theta}\frac{\theta}{\kappa}$ *and* $\kappa(\theta, u) = F^{-1}(\theta u / M)$, *then there exists a unique equilibrium if for all* (u, θ),

$$\frac{\beta}{1-\eta} \left| 1 - \delta - \gamma p(\theta) \left(1 + \frac{\varepsilon_{p,\theta}}{\varepsilon_{q,\theta} + \varepsilon_{\kappa,\theta}} \right) \right| < 1. \tag{15}$$

In particular, within the class of mean preserving spreads F_{σ_κ} *with standard deviation* σ_κ *of some given distribution, condition* (14) *and* (15) *are satisfied for* σ_κ *large.*

Proposition 2 follows the statement of Proposition 1. The conditions for uniqueness are, however, much weaker than in the absence of heterogeneity. For instance, condition (14), which mirrors condition (12), displays one additional term that involves the elasticity of κ with respect to θ. Consistent with our previous interpretation of this condition, the left hand side should be understood as measuring the strength of the complementarity, while the right hand side captures the degree of substitutability. The additional term $\varepsilon_{\kappa,\theta} > 0$ results purely from the introduction of heterogeneity and corresponds to term (3) in equation (11). With heterogeneity in κ, an increase in θ requires to climb up the distribution of costs, promoting uniqueness by making vacancy posting less attractive when too many firms enter. As a result, there are more forces contributing to substitutability between agents and condition (14) is easier to satisfy with heterogeneity than without. Equation (15) is the condition for a contraction. In comparison to (13), this condition includes a new term involving the elasticity $\varepsilon_{\kappa,\theta}$ that relates to heterogeneity, which we interpret as follows. When heterogeneity increases, the marginal cost $\hat{\kappa}$ must vary much more to accommodate a change in tightness θ. As a result, the elasticity term $\varepsilon_{\kappa,\theta}$ goes to ∞ when $\sigma_\kappa \to \infty$, making the complementarity in demand weaker in the face of other forces (η low) and allowing both conditions (14) and (15) to be satisfied. As Proposition 2 shows, the large response in marginal costs required by a change in the economy tends to slow down the reactivity of θ, thereby reducing the strength of the complementarity and allowing for uniqueness when heterogeneity is large.

Non-linear dynamics

After having demonstrated that the presence of heterogeneity may avoid multiplicity, we now argue that the model still retains interesting dynamic implications, even when uniqueness obtains, owing to the presence of strong complementarities.

Figure 3 describes how the entry decision is affected by shocks in an economy without demand linkages (panel (a)) and an economy with complementarities in demand but a sufficient dispersion in vacancy costs to deliver uniqueness. In both instances, the function Ψ shifts up when the fundamentals of the economy improve (higher z, lower u), leading to more entry. In the absence of complementarities, $\sigma = \infty$, the inflow of new firms is quickly inhibited by the crowding out and higher marginal cost effects. With complementarities, however, the equilibrium marginal cost $\hat{\kappa}$ may experience large jumps in the region where Ψ flattens. As a result, the response of unemployment to various shocks may be greatly amplified, leading to more volatility and additional propagation.

From this figure, we conclude that the equilibrium marginal $\hat{\kappa}$ is high when productivity z is high or unemployment u is small, implying a high rate of job creation. When productivity is low or unemployment is high, on the other hand, the marginal cost $\hat{\kappa}$ is low, leading to a low market tightness and a low rate of job creation. Figure 4 illustrates how the law of motion for unemployment looks like in our economy for three different values of productivity z. In line with the above discussion, job creation is high for low values of unemployment, leading to a rather low future unemployment rate, but the rate of job creation may experience a significant jump as the unemployment rate increases. This jump, which corresponds to the S-shaped area in the law of motion for u, is when a major part of the distribution of entrants shifts from posting to not posting vacancies – when aggregate demand is so low that firms find it unattractive to post vacancies. For low values of z, this shift takes place early, for low values of unemployment, while it may never be observed for high levels of productivity. From this particular non-linear law of motion, a good steady state generally exists at low values for unemployment, but a bad steady state may sometimes appear in the high unemployment region when productivity is really low. As a consequence, the economy may sometimes experience large unemployment crises when this high steady state is the only attractor in this dynamic system.

We thus conclude that, when the degree of heterogeneity is high enough, the unique equilibrium of this economy displays interesting dynamic properties, including non-linear responses to shocks and the possibility of multiple stationary unemployment rates for a given level of productivity. We explore these features in the calibrated economy of the next section.

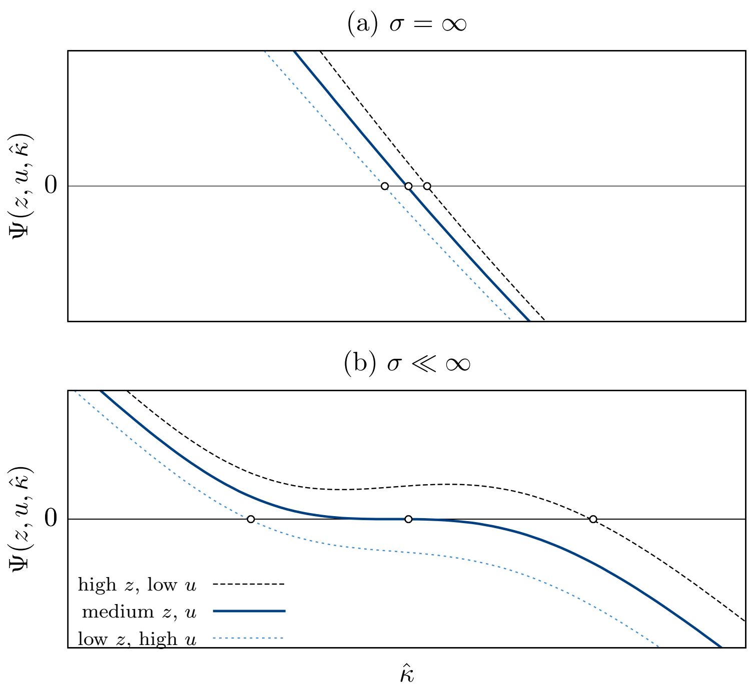


Figure 3: Large non-linearities with complementarities

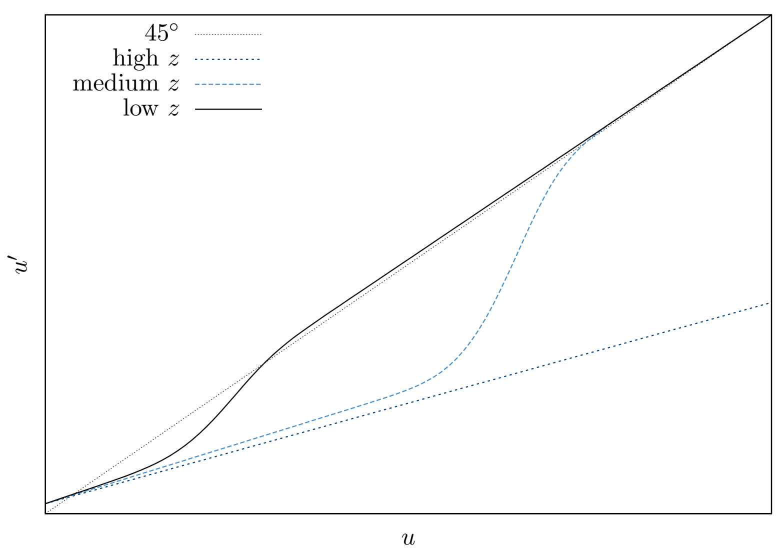


Figure 4: Steady state shifts in the dynamics of u

4 Quantitative Exercises

In this section, we first calibrate the model and explore to what degree it can explain the cyclical properties of labor market aggregates. We then evaluate the amount of volatility and propagation generated by the aggregate demand channel. Finally, we explore the model's non-linear properties and its ability to generate deep prolonged recessions.

4.1 Calibration

We begin by calibrating the model to U.S. data. Our sample starts in 1951, the first year the Conference Board help wanted data is available and ends in 2015. We detrend all time series linearly to preserve the autocorrelation structure of the data.[4] We follow Hagedorn and Manovskii (2008) in setting the time period to one-twelfth of a quarter, which corresponds to about a week. Since the data we use is primarily available at a quarterly frequency, we aggregate all time series generated by the model before comparing them to the data.

As in Shimer (2005), we set the discount rate to $\beta = 0.988^{1/12}$. We normalize the constant part of the productivity, A, so that a firm's revenue is 1 at the steady state.[5] We set $s = 0.09$ to match the average fraction of the labor force self-employed in our sample. Hagedorn and Manovskii (2008) estimate the monthly separation rate to be 0.026. Converting to weekly frequency, we set $\delta = 0.0081$.[6] We adopt the matching function of Den Haan et al. (2000) so that $q(\theta) = (1 + \theta^{\mu})^{-1/\mu}$ and $p(\theta) = \theta q(\theta)$. Den Haan et al. (2000) find an average *monthly* job filling rate of $\bar{q}_m = 0.71$ and Shimer (2005) estimates the average *monthly* job finding rate to be $\bar{p}_m = 0.45$. Using these values, we find a steady-state labor market tightness of $\bar{\theta} = \bar{p}_m/\bar{q}_m = 0.63$. To have the market tightness $\bar{\theta}$ consistent with the *weekly* job finding rate $\bar{p} = 0.139$ requires setting $\mu = 0.4$.

We set the elasticity of substitution parameter $\sigma = 4$ as our benchmark. This number is in line with many plant-level estimates. Bernard et al. (2003) find a value of $\sigma = 3.79$ in a model of plant-level export. Similarly, Broda and Weinstein (2006) estimate an elasticity of $\sigma = 3$ at various levels of aggregation. Among macroeconomic studies Christiano et al. (2015) estimate a New-Keynesian model with financial frictions and find an elasticity of 3.78. Small values of σ are often rejected as implying abnormally high markups but this is not the case in our model. In our calibrated economy, for instance, markups are at the low level of 2% on average.[7]

We estimate the properties of the productivity process z using equation (10) together with data

[4] See King and Rebelo (1993) for a discussion of some drawbacks of the HP filter. Fajgelbaum et al. (2015) show that the HP filter substantially alters the properties of persistent recessions.

[5] We set $A = (1 - \bar{u})^{-\frac{1}{\sigma - 1}}$, where $\bar{u}$ is the steady-state level of unemployment. This normalization implies that changes in σ only affect the cyclicality of revenues and not their level.

[6] See Hagedorn and Manovskii (2008) for the steps of the conversion from monthly to weekly.

[7] In the model, we have that Markup $= \frac{\text{Unit price}}{\text{Unit cost}} = \frac{P_j}{w/Y_j} = \frac{P_j Y_j}{\gamma P_j Y_j + (1-\gamma) b + \gamma \beta \theta \hat{\kappa}}$. Since firm revenues $P_j Y_j$ are normalized to 1 and that the calibration targets the steady-state values of $\hat{k}$ and θ from the data, we see that σ has no influence on markups at the steady state.

on output per worker and the unemployment rate. We find a standard deviation of the ergodic distribution of z of 5% and a quarterly autocorrelation of 0.984.

For the distribution of the cost of posting a vacancy, we rely on data gathered by Abowd and Kramarz (2003). They combine three datasets to estimate the costs of hiring in French firms. They find that the average cost of *hiring* a worker, which we denote k, was 5,560 French Francs in 1992, with a standard deviation of 26,240. The average cost of labor, analogous to the wage w in our model, was 171,022 per year with a standard deviation of 676,185. We assume that the cost of hiring a worker is proportional to the wage, $k = Dw$, where the cost per unit of wage, D, is iid. Using the moments reported by Abowd and Kramarz (2003) we find that $E(D) = 0.0325$ and $std(D) = 0.0205$, meaning that hiring one worker costs about 3.25% of the annual wage. In the model, hiring one worker costs an average of $E(\kappa|\kappa < \hat{\kappa})/q(\bar{\theta})$ so that

$$0.0325 \times (48 \times w) = \frac{E(\kappa|\kappa < \hat{\kappa})}{q(\bar{\theta})}$$

where w denote the wage over a twelfth of a quarter, which is approximately 1 in our simulations. Using the steady-state value $\bar{q} = 0.22$ we find that the average paid vacancy posting cost is $E(\kappa|\kappa < \hat{\kappa}) = 0.34$. A similar calculation gives $std(\kappa|\kappa < \hat{\kappa}) = 0.21$.[8]

We use steady-state versions of the Bellman equation (5) and the free-entry condition (9) to find the steady-state value of $\hat{\kappa}$ as a function of the bargaining power γ and the value of leisure b

$$q(\bar{\theta})\,\beta\frac{(1-\gamma)(1-b)}{1-\beta\left(1-\delta-\gamma p(\bar{\theta})\right)} = \hat{\kappa}.$$

Assuming that the costs κ are normally distributed, we can use the values of $E(\kappa|\kappa < \hat{\kappa})$ and $std(\kappa|\kappa < \hat{\kappa})$ to fully characterize the distribution F. Finally, we can find M since, at the steady-state, $\bar{v} = MF(\hat{\kappa})$ and $\bar{v} = \bar{\theta} \times \bar{u} = 0.035$ in the data.

We are now left with γ and b to parametrize. We do so by targeting an average unemployment rate of 5.5% and an elasticity of wages to productivity of 0.8, which is the target suggested by Haefke et al. (2013) to calibrate search models. We find a bargaining power of $\gamma = 0.2725$ and $b = 0.8325$. Both of these numbers are well within the range of parameters used in the literature.[9]

Table 1 summarizes the parameter values of the calibrated economy.

We verify that these parameters imply a unique equilibrium. To do so we iterate from the theoretical lower and upper bounds and check that both series of iterations converge to the same

[8] Hall and Milgrom (2008) use numbers from Silva and Toledo (2009) who find that recruitment costs are 14% of quarterly pay per hire. In our setup this would amount to $\kappa = 0.37$, not far from our own estimate of 0.34.

[9] For instance, Hagedorn and Manovskii (2008) use $\gamma = 0.052$ and $b = 0.955$ while Shimer (2005) uses $\gamma = 0.72$ and $b = 0.4$.

Parameter	Value
Time discount	$\beta = 0.988^{1/12}$
Steady-state productivity	$A = (1-\bar{u})^{-1/(\sigma-1)}$
Persistence of productivity	$\rho_z = 0.984^{1/12}$
Precision of innovations to productivity	$\gamma_z = \left(0.05^2 \times \left(1-\rho_z^2\right)\right)^{-1}$
Elasticity of substitution between goods	$\sigma = 4$
Fraction of self-employed workers	$s = 0.09$
Job destruction rate	$\delta = 0.0081$
Parameter of matching function	$\mu = 0.4$
Mean of $F(\kappa)$	$E(\kappa) = 2.12$
Standard deviation of $F(\kappa)$	$std(\kappa) = 0.67$
Mass of potential entrants	$M = 3.29$
Worker bargaining power	$\gamma = 0.2725$
Value of non-work activities	$b = 0.8325$

Table 1: Parameters of the calibrated economy

equilibrium within numerical tolerance.[10]

4.2 Dynamics

We first consider how the aggregate demand channel influences the dynamic properties of the economy. To so, we plot in Figure 5 the *change* in the unemployment rate, $\Delta u_t = u_{t+1} - u_t$, as a function of the unemployment rate u_t for various levels of productivity z. The thick blue curves represents the dynamics of the unemployment rate in the calibrated economy at the steady-state level of z (squares), a low level of z (circles) and a very low level of z (triangles).[11] The dashed brown curves represent the same corresponding dynamics when there are no demand complementarities between firms ($\sigma = \infty$).[12]

Consider first the dynamics of unemployment at the steady-state level of z (squares). We see on Figure 5 that, in this case, both models behave in a similar way. There is a unique steady-state level of unemployment $\bar{u}$ around 5% and both models would converge to it fairly quickly without shocks to z. Above the steady state, there is an abundance of unemployed workers and vacancies are filled quickly, which incentivizes firms to hire massively. As a result, the unemployment rate declines quickly. The opposite logic operates to push the economy towards higher unemployment rate when $u_t < \bar{u}$. In the neighborhood of the steady state, the local dynamics of both models is

[10]Lemma 5 in the appendix provides a condition under which the mapping that characterizes the equilibrium is a monotone operator. This condition, which is satisfied in our calibration, ensures that this procedure to check uniqueness is valid. Indeed, starting the iteration procedure from the upper (resp. lower) bound of the space in which our value function lies is guaranteed to converge to the maximal (resp. minimal) fixed point. That both the maximal and minimal fixed points coincide ensures that the fixed point is unique.

[11]The steady-state level corresponds to $z = 0$ while the low and very low levels corresponds to -2 and -2.5 standard deviations of the ergodic distribution of z.

[12]The normalization $\bar{A}$ implies that changing σ does not affect the steady-state level of revenues.

identical. Indeed, since steady state profits are the same in both models, and that, in expectations, firms expect to remain around the steady state next period, the incentives to post vacancies are the same and so is the dynamics of u.

The dynamics of the two models start to differ for lower values of z. For the low z (circles in Figure 5), the two models have very similar steady states but their behavior away from the steady state is very different. In the model without complementarity, the further away the economy is from its steady state, the fastest it will move towards it. The same is not true in the model with complementarity. Indeed, as the unemployment rate increases, firms expect to have lower demand for their products and, therefore, hire fewer workers. This force slows the economy's recovery and can create additional steady states in the dynamics of u_t. For the low z, three such steady states exists: 5%, 35% and 50%. At these points, the traditional crowding out effect of the MP model exactly offsets the demand complementarity to keep the unemployment rate at a constant level. While the 35% and 50% steady states exhibit very high unemployment, compared to the U.S. data, their presence is the consequence of forces that affect the economy at lower levels of unemployment.[13]

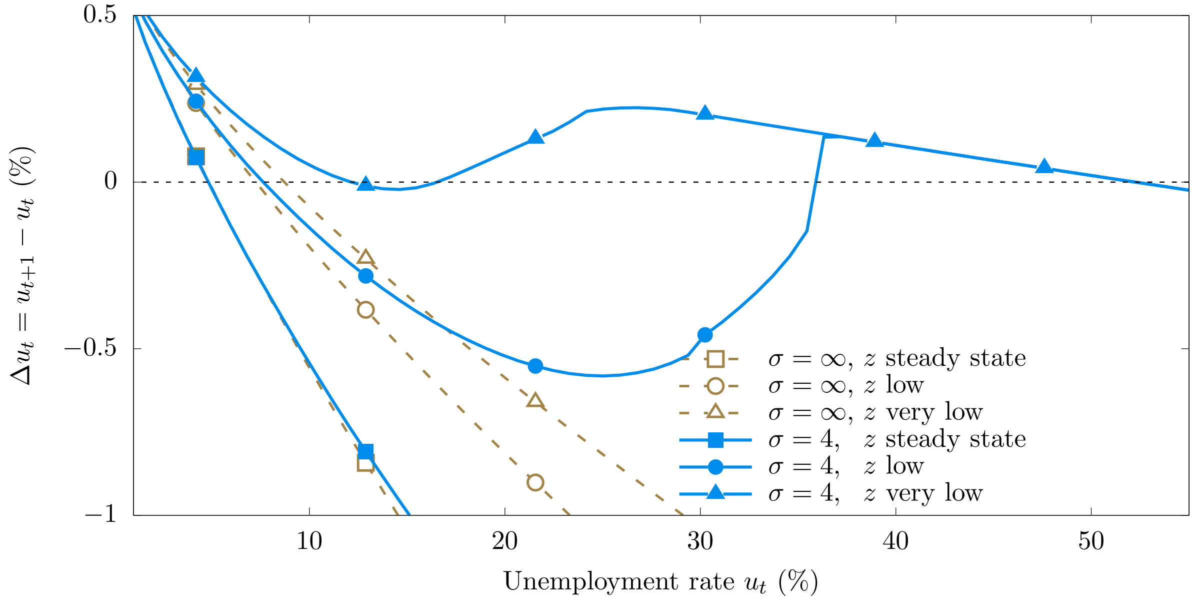
Finally, for very low productivity (triangles in Figure 5), the demand externality is strong enough to influence the dynamics of the economy even for low unemployment levels. In this case, firms anticipate that future unemployment rates will be high and therefore that future aggregate demand will be low, which lowers the incentive to post vacancies.

4.3 Response to shocks

Figure 5 highlights the race between the counteracting forces at work in our model: the substitutabilities due to crowding out and marginal cost effects, and the complementarity in demand. The differences between the two versions of the model highlights that the demand externality has a stronger impact on the economy when productivity is low and unemployment is high. As a result, we expect that the dynamics of our model will differ the most from the MP model after large or long-lasting shocks. We therefore consider how the economy reacts to shocks of different magnitudes.

We first consider the impact of a small shock, depicted in Figure 6. Panel (a) features the productivity process that is fed into the model and the others panels show the response of the economy. All time series are aggregated to a quarterly frequency. The solid curves correspond to the full model while the dashed curves correspond to an economy with the same parameters except that there is no complementarity ($\sigma = \infty$). We see that both economies respond very similarly to the shock. This is not surprising since the complementarity acts through changes in unemployment,

[13] Sterk (2015) looks for the presence of multiple steady states in the dynamics of unemployment over the last 25 years in the United States economy and finds that it features a stable steady state around 5% unemployment and an unstable steady state around 10% unemployment.



Notes: The steady-state level corresponds to $z = 0$ while the low and very low levels corresponds to -2 and -2.5 standard deviations of the ergodic distribution of z.

Figure 5: Dynamic of the unemployment rate

a slow moving variables that does not respond much to small shocks.

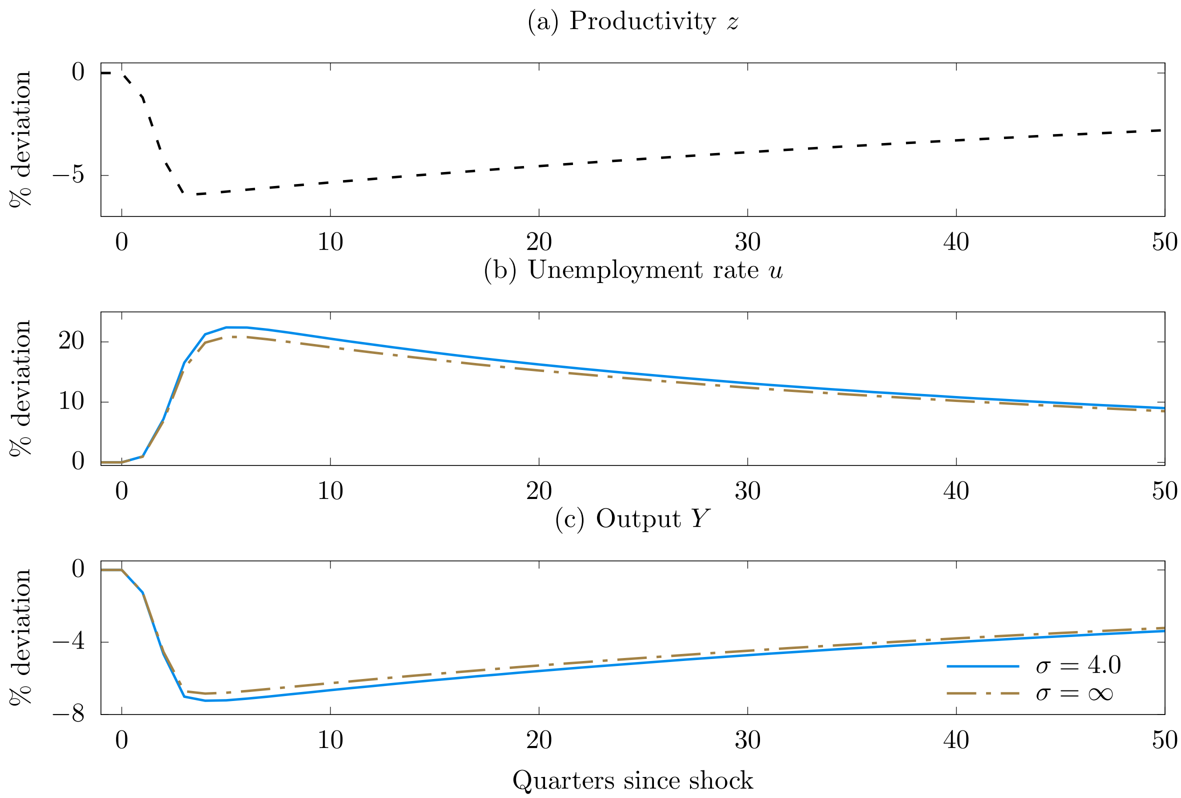
In contrast, Figure 7 shows how both economies react to a large shock. Their behaviors are very different. In this case, since the shock is large and long-lasting, the unemployment rate begins to rise substantially. As a result, aggregate demand declines, which lowers the incentives for firms to post vacancies and raises unemployment further up. We see on the figure that the mechanism *amplifies* and *propagates* the shock. In terms of amplification, the overall decline in output is about twice as large when the aggregate demand channel is active and the unemployment rate is multiplied by four . In terms of propagation, the trough in output and the peak in unemployment happen after 11 quarters instead of 5.

Figures 6 and 7 highlight the nonlinear nature of the aggregate demand mechanism. For small shocks, our economy is barely distinguishable from the standard MP model, while for large shocks the differences are sizable.

4.4 Cyclical Properties

We now investigate whether the dynamic properties of the model could help explain the U.S. data.

Table 2 presents summary statistics of U.S. data from the first quarter of 1951 to the first quarter of 2015 along with the corresponding numbers in a long simulation of the calibrated economy ($\sigma = 4$). The last rows of Table 2 present the same numbers in a long simulation of an economy



Notes: The innovation to z is set to -1 standard deviation for 2 quarters. The simulated data is aggregated from weekly to quarterly.

Figure 6: Impact of a small shock

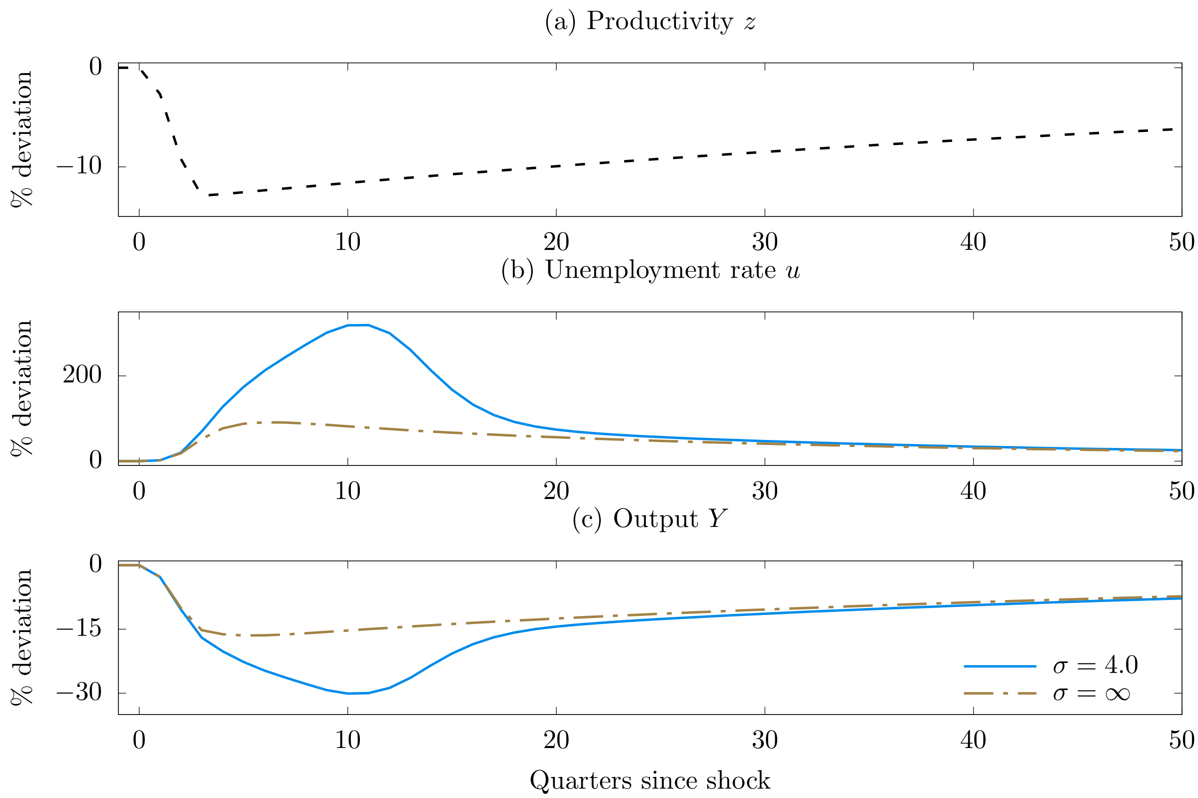
with no complementarities ($\sigma = \infty$). We see that the benchmark model generates an amount of variation in unemployment u and vacancies v roughly in line with the data while it produces too much volatility in labor market tightness θ. The model with no complementarities, on the other hand, generates much less variations in these aggregates.

In terms of autocorrelations, both models perform similarly and generate too much persistence compared to the data. Both models are also similar in term of the correlations between u, v and θ. While all the correlations have the same sign as in the data, they are much stronger in the two models, a consequence of having a single shock driving the dynamics.[14]

Overall, Table 2 shows that including standard demand linkages in an MP model leads to additional volatility in labor market aggregates.

Demand linkages also generate a quantitatively significant propagation channel. To see this, we consider in Figure 8 the autocorrelograms of productivity growth, output growth and the growth in labor market tightness in the data, our benchmark model and a model with no demand complemen-

[14] In particular, as is common in MP models, output per worker is much less correlated with labor market aggregates in the data than in the model. In the data, the labor market tightness reacts sluggishly to productivity shocks. Fujita and Ramey (2007) show how a search model with sunk costs for vacancy creations can replicate this features while lowering the contemporaneous correlation between output per workers and various labor market aggregates.



Notes: The innovation to z is set to -2.3 standard deviations for 2 quarters. The simulated data is aggregated from weekly to quarterly.

Figure 7: Impact of a large shock

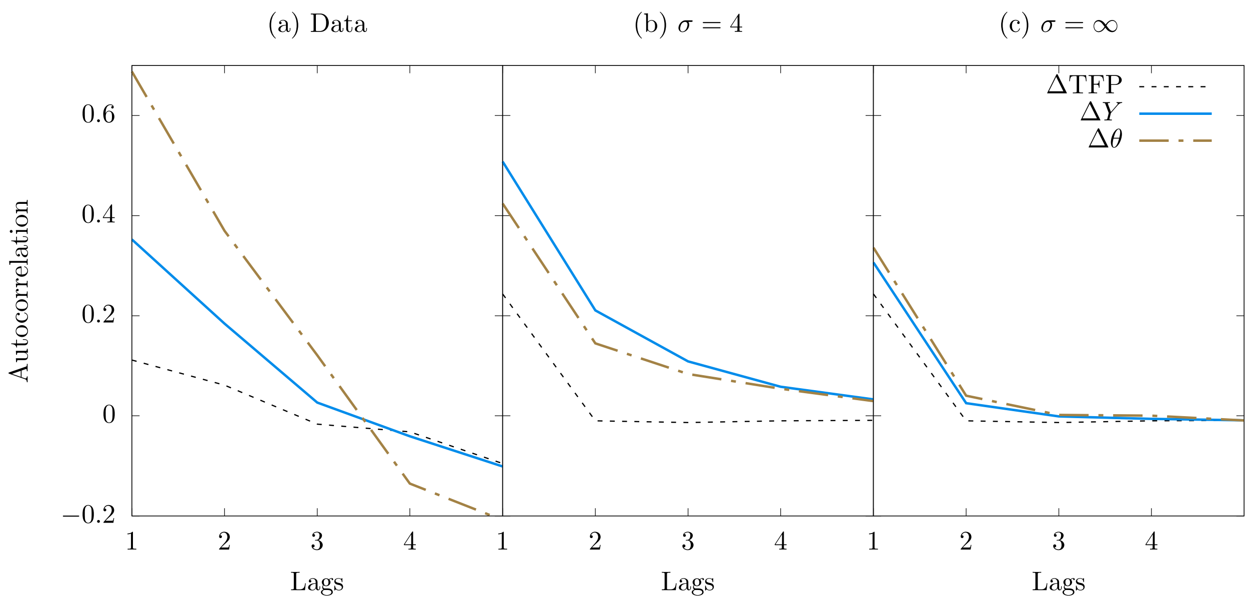
tarities. In the data, output growth and the growth in labor market tightness are more persistent than productivity growth, a fact that the model without demand complementarity struggles to replicate because it features weak propagation in general. We see, however, that the mechanism makes output growth significantly more persistent than productivity growth. Similarly, the mechanism increases the persistence of the growth in the labor market tightness.

We conclude from these exercises that including aggregate demand linkages in the standard search model generates quantitatively significant increases in the volatility and persistence of labor market aggregates, thereby bringing the model closer to the data.

		u	v	θ
(a) Data				
Standard deviation		0.260	0.289	0.439
Quarterly autocorrelation		0.969	0.976	0.962
	u	1	-0.276	-0.774
Correlation matrix	v	—	1	0.822
	θ	—	—	1
(b) Benchmark model ($\sigma = 4$)				
Standard deviation		0.280	0.251	0.527
Quarterly autocorrelation		0.996	0.985	0.993
	u	1	-0.970	-0.993
Correlation matrix	v	—	1	0.992
	θ	—	—	1
(c) No complementarities ($\sigma = \infty$)				
Standard deviation		0.162	0.149	0.309
Quarterly autocorrelation		0.994	0.982	0.991
	u	1	-0.983	-0.996
Correlation matrix	v	—	1	0.995
	θ	—	—	1

Notes: Quarterly U.S. data from 1951:Q1 to 2015:Q1. All variables are seasonally adjusted. Variables observed monthly are averaged over quarters. All variables are in logs and linearly detrended. Unemployment rate is constructed by dividing Unemployment (UNEMPLOY) by the Civilian Labor Force (CLF16OV), both from the Bureau of Labor Statistics. The vacancy rate is constructed by merging the Conference Board Help-Wanted index with the Job Openings and Labor Turnover Survey data on Job Opening: Total Nonfarm (JTSJOL). The models are simulated for one million periods. All model time series are aggregated to a quarterly frequency.

Table 2: Cyclical properties of labor market aggregates



Notes: The figure shows the autocorrelogram of the growth rate of the productivity, output Y and the labor market tightness θ.

Figure 8: The impact of the mechanism on the propagation of shocks

5 Conclusion

We introduce monopolistic competition into an otherwise standard search and matching model of the labor market. With this extension, the general level of aggregate demand matters for firms' hiring decisions. In a downturn, an increase in unemployment leads to a decline in aggregate demand that further depresses labor market variables.

This aggregate demand externality creates a coordination problem that leads to multiple equilibria. We show, however, that the multiplicity result is fragile and sensitive to the introduction of heterogeneity across agents. Some heterogeneity in vacancy costs suffices to restore uniqueness. Despite the absence of multiple equilibria, the model retains interesting properties including a highly non-linear response of job creation to shocks that can lead to multiplicity of steady states in unemployment.

We calibrate the model to the U.S. economy and show that the aggregate demand channel improves on the standard model by generating more volatility and persistence in labor market variables. The model can also generate deep unemployment crises after large enough shocks.

To preserve the transparency of the argument and allow for comparison with the literature, we have kept the model simple. Several extensions would be worth investigating. First, allowing for savings or capital in the model could affect the strength of the aggregate demand channel by letting people smooth consumption over time. Second, the introduction of price and wage rigidities could further magnify the type of dynamics that we describe by amplifying the role of demand linkages. Third, because of the aggregate demand externality and the bargaining assumption, the model is generally inefficient, offering a role for government intervention. We leave these topics to future research.

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A Proofs

This section contains the proofs of the propositions stated in the paper. We proceed with the most general model, which includes demand linkages and heterogeneity in vacancy costs, and then specialize the results to specific cases.

A.1 Notation

In what follows, we denote $B(X)$ the Banach space of bounded continuous functions $f : X \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ for some $n \in \mathbb{N}^*$, equipped with the sup norm $\|f\| = \sup_{x\in X} |f(x)|$. We also denote $\varepsilon_{Y,X} = \left|\frac{d\log Y}{d\log X}\right|$ the absolute value of the elasticity of variable or function Y with respect to variable X, keeping other things equal. Throughout the proofs, we use, in particular, the following elasticities:

$$\varepsilon_{q,\theta} = -\frac{q'(\theta)\theta}{q(\theta)},\ \varepsilon_{p,\theta} = \frac{p'(\theta)\theta}{p(\theta)} = 1 - \varepsilon_{q,\theta},\ \varepsilon_{\kappa,\theta} = \varepsilon_{\kappa,u} = \frac{\theta u}{MF'\left(F^{-1}\left(\frac{\theta u}{M}\right)\right)}\frac{1}{F^{-1}\left(\frac{\theta u}{M}\right)}.$$

The notation $\overline{\varepsilon}_{Y,X}$ (resp. $\underline{\varepsilon}_{Y,X}$) denotes the lowest upper (resp. highest lower) bound on this elasticity.

A.2 Problem statement

The value function of firms is given by

$$J(z,u) = (1-\gamma)\left(Ae^z(1-u)^{\frac{1}{\sigma-1}} - b\right) + \beta(1-\delta-\gamma p(\theta))E\left[J\left(z',u'\right)|z\right]$$

subject to

$$u' = \delta(1-s) + u(1-\delta-p(\theta)).$$

The labor market tightness $\theta = \frac{M}{u}F(\hat{\kappa})$ solves the free entry problem with $\theta(z,u) \in \Theta(z,u)$, where

$$\begin{aligned}\Theta(z,u) = &\left\{\theta \in \left[0, \frac{M}{u}\right] \mid q(\theta)\beta EJ\left(z', \delta(1-s) + u(1-\delta-p(\theta))\right) = F^{-1}\left(\frac{\theta u}{M}\right)\right\}\\ &\cup\left\{0 \text{ if } q(0)\beta EJ\left(z',\delta(1-s)+u(1-\delta-p(0))\right) < \underline{\kappa}\right\}\\ &\cup\left\{\frac{M}{u} \text{ if } q\left(\frac{M}{u}\right)\beta EJ\left(z',\delta(1-s)+u\left(1-\delta-p\left(\frac{M}{u}\right)\right)\right) > \overline{\kappa}\right\}.\end{aligned}$$

A.3 Assumptions

Assumption 1. *Vacancy costs are distributed according to the cumulative distribution function F with support $(\underline{\kappa},\overline{\kappa}) \in \left(\mathbb{R}_+^* \cup \{\infty\}\right)^2$ with $\underline{\kappa} \leqslant \overline{\kappa}$. F has mean μ_κ and standard deviation σ_κ.*

Assumption 2. *Aggregate productivity z follows the truncated autoregressive process*

$$z' = max\{min\ [\rho z + \varepsilon_z, \overline{z}], \underline{z}\},$$

where $\underline{z} \leqslant \overline{z}$ and $\varepsilon_z \sim \mathcal{N}\left(0, \left(1-\rho^2\right)\sigma_z^2\right)$.

The following assumption guarantees that the market tightness $\theta(z,u)$ never exceeds some upper bound θ_{max} such that $p(\theta_{max}) \leqslant 1-\delta$. It ensures also that u is bounded below by $\delta(1-s)$.

Assumption 3. *The parameters are such that there exists $\theta_{max} \in \mathbb{R}_+$ such that $p(\theta_{max}) \leqslant 1-\delta$ and $\beta\overline{J} < \frac{F^{-1}\left(\frac{\theta_{max}\delta(1-s)}{M}\right)}{q(\theta_{max})}$, where $\overline{J}$ is defined in Definition 2. Denote $\kappa_{max} = min\left[F^{-1}\left(\frac{\theta_{max}(1-s)}{M}\right), \overline{\kappa}\right]$.*

Assumption 4. *The firm matching probability $q(\theta)$ is strictly decreasing, continuously differentiable and strictly positive over $[0,\theta_{max}]$. The worker matching probability $p(\theta) = \theta q(\theta)$ is strictly increasing.*

Assumption 5. *The parameters are such that $Ae^{\underline{z}}s^{\frac{1}{\sigma-1}} > b$ and $\gamma < 1-\delta$.*

A.4 Definitions

Definition 2. Let $\Omega = [\underline{z},\overline{z}] \times [\delta(1-s), 1-s]$ and $\mathcal{J} \subset B(\Omega)$ the set of bounded continuous function such that for all $J : (z,u) \in \Omega \to \mathbb{R} \in \mathcal{J}$, i) $J(z,u)$ belongs to $\left[\underline{J}, \overline{J}\right]$ where $\underline{J} = \frac{1}{1-\beta(1-\delta-\gamma p(\theta_{max}))}(1-\gamma)\left(Ae^{\underline{z}}s^{\frac{1}{\sigma-1}} - b\right)$ and $\overline{J} = \frac{1}{1-\beta(1-\delta)}(1-\gamma)\left(Ae^{\overline{z}} - b\right)$, ii) J is Lipschitz continuous in u with modulus $\overline{J}_u = (1-\beta)^{-1}\left[(1-\gamma)Ae^{\overline{z}}\frac{s^{-\frac{\sigma}{\sigma-1}}}{\sigma-1} + \frac{\beta\gamma\overline{J}(1-\delta)}{(1-\eta)\delta(1-s)}\right]$, i.e., for all $z \in [\underline{z},\overline{z}]$ and $(u_1,u_2) \in [0,1-s]^2$ such that $u_1 \leqslant u_2$,

$$|J(z,u_2) - J(z,u_1)| \leqslant \overline{J}_u |u_2 - u_1|.$$

Definition 3. Let $\Psi : (J,z,u,\theta) \in B(\Omega) \times [\underline{z},\overline{z}] \times [0,1-s] \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ the function such that

$$\Psi(J,z,u,\theta) = \beta EJ\left(z', \delta(1-s) + u(1-\delta-p(\theta))\right) - \frac{\kappa(\theta,u)}{q(\theta)},$$

where $\kappa(\theta,u) = F^{-1}\left(\frac{\theta u}{M}\right)$.

Definition 4. For $J \in B([\underline{z},\overline{z}] \times [0,1-s])$, let $\Theta(J) : [\underline{z},\overline{z}] \times [0,1-s] \rightrightarrows \mathbb{R}_+$ be the correspondence such that

$$[\Theta(J)](z,u) = \left\{\theta \in \left[0, \frac{M}{u}\right] \mid \Psi(J,z,u,\theta) = 0\right\} \cup \{0 \text{ if } \Psi(J,z,u,0) < 0\}$$
$$\cup \left\{\frac{M}{u} \text{ if } \Psi\left(J,z,u,\frac{M}{u}\right) > 0\right\}.$$

Definition 5. Let $J \in B(\Omega)$ and $\theta(z,u)$ a particular selection of solutions to the free entry condition, that is to say, for all $(z,u) \in \Omega$, $\theta(z,u) \in [\Theta(J)](z,u)$. We define the mapping $T_\theta : J \in B([\underline{z},\overline{z}] \times [0,1-s]) \longrightarrow B([\underline{z},\overline{z}] \times [0,1-s])$ such that

$$[T_\theta(J)](z,u) = (1-\gamma)\left(Ae^z(1-u)^{\frac{1}{\sigma-1}} - b\right) + \beta(1-\delta-\gamma p(\theta))E\left[J\left(z',u'\right)|z\right]$$

where $u' = \delta(1-s) + u(1-\delta-p(\theta))$. When $\theta(z,u)$ is a singleton for all $(z,u) \in \Omega$, the mapping $T_\theta(J)$ is uniquely determined and we denote it by $T(J)$.

A.5 Propositions

Proposition 1 (Full)**.** *Suppose Assumptions 1-5 are satisfied and there is no heterogeneity in vacancy costs. Then, we have the following:*

1. *In the case without aggregate demand externality, $\sigma = \infty$, the equilibrium exists and is unique.*

2. *In the case with $\sigma < \infty$, assuming that there exists $0 < \eta < 1-(1-\delta)^2$ such that for all $u \in [\delta(1-s), 1-s]$ and $\theta \in [0, \theta_{max}]$,*

$$\beta \overline{J}_u u p(\theta)\varepsilon_{p,\theta} \leqslant \eta \frac{\kappa(\theta,u)}{q(\theta)}\varepsilon_{q,\theta}, \tag{12}$$

where $\overline{J}_u = (1-\beta)^{-1}\left[(1-\gamma)Ae^{\overline{z}}\frac{s^{-\frac{\sigma}{\sigma-1}}}{\sigma-1} + \beta\frac{\eta}{1-\eta}\frac{1-\delta}{\delta}\frac{\gamma}{1-s}\overline{J}\right]$ and $\overline{J} = \frac{1}{1-\beta(1-\delta)}(1-\gamma)\left(Ae^{\overline{z}} - b\right)$, then there exists a unique equilibrium if

$$\frac{\beta}{1-\eta} max_{\theta \in [0,\theta_{max}]}\left|1-\delta-\gamma p(\theta)\left(1+\frac{\varepsilon_{p,\theta}}{\varepsilon_{q,\theta}}\right)\right| < 1. \tag{13}$$

Proof. Without heterogeneity in fixed costs, the distribution F is degenerate at one point that we denote $\kappa = \underline{\kappa} = \overline{\kappa}$. In particular, the elasticity $\varepsilon_{\kappa,\theta}$ is null.

1. In the case without aggregate demand externality, the model is that standard Diamond-Mortensen-Pissarides model and the value of the firm is

$$J(z) = (1-\gamma)(Ae^z - b) + \beta(1-\delta-\gamma p(\theta))E\left[J\left(z'\right)|z\right],$$

while the labor market tightness is a jump variable determined by the free entry condition

$$\frac{\kappa}{q(\theta)} = \beta E\left[J\left(z'\right)\right].$$

In this particular case, the value function and the market tightness are independent of the unemployment rate u. We thus look for an equilibrium in the space $\mathcal{J}$ (Definition 2) with $\overline{J}_u = 0$. In that case, Lemma 2 establishes that the free entry condition admits one and exactly one solution. Lemma 4 shows that the equilibrium mapping $T(J)$ that characterizes the economy is a well-defined self-map on $\mathcal{J}$. In particular, with the assumption that the $J \in \mathcal{J}$ is such that its modulus of Lipschitz continuity is $\overline{J}_u = 0$, we obtain $\eta = 0$ in condition (17), and the modulus of Lipschitz continuity for $T(J)$ is

$$(1-\gamma) A e^{\overline{z}} \frac{s^{-\frac{\sigma}{\sigma-1}}}{\sigma-1} + \frac{\beta\gamma\overline{J}(1-\delta)}{(1-\eta)\delta(1-s)}\left[\eta + (1-\eta)\frac{\varepsilon_{\kappa,\theta}}{\varepsilon_{q,\theta}+\varepsilon_{\kappa,\theta}}\right] \underset{\substack{\sigma\to\infty \\ \eta=0,\varepsilon_{\kappa,\theta}=0}}{\longrightarrow} 0.$$

Lemma 6 finally demonstrates that the mapping T is a contraction on the Banach space $\mathcal{J}$ if for all $\theta \in [0, \theta_{max}]$,

$$\frac{\beta}{1-\eta}\left|1-\delta-\gamma p(\theta)\left(1+\frac{\varepsilon_{p,\theta}}{\varepsilon_{q,\theta}+\varepsilon_{\kappa,\theta}}\right)\right| < 1,$$

which, in the current case where $\eta = 0$ and $\varepsilon_{\kappa,\theta} = 0$, simplifies to $\beta(1-\delta-\gamma p(\theta))$, obviously satisfied.

2. In the case with an aggregate demand externality, the condition for existence and uniqueness are less easily satisfied. In particular, the value function now depends on the unemployment rate and $\overline{J}_u = (1-\beta)^{-1}\left[(1-\gamma) A e^{\overline{z}} \frac{s^{-\frac{\sigma}{\sigma-1}}}{\sigma-1} + \frac{\beta\gamma\overline{J}(1-\delta)}{(1-\eta)\delta(1-s)}\right]$. Under condition (12) with $\eta < 1$, Lemma 2 tells us that the free entry problem admits a unique solution. Lemma 4 and 6 then establish that the mapping $T(J)$ that characterizes the equilibrium is a contraction on $\mathcal{J}$ if

$$\forall\theta \in [0, \theta_{max}], \ \frac{\beta}{1-\eta}\left|1-\delta-\gamma p(\theta_\lambda)\left(1+\frac{\varepsilon_{p,\theta}}{\varepsilon_{q,\theta}+\varepsilon_{\kappa,\theta}}\right)\right| < 1,$$

which, imposing $\varepsilon_{\kappa,\theta} = 0$ and $\varepsilon_{p,\theta} = 1 - \varepsilon_{q,\theta}$ and evaluating for all θ, can be summarized by

$$\frac{\beta}{1-\eta}\max_{\theta\in[0,\theta_{max}]}\left|1-\delta-\gamma\frac{p(\theta)}{\varepsilon_{q,\theta}}\right| < 1.$$

As this equation shows, this condition can only be satisfied for η small, implying that the complementarity due to the demand linkages cannot be too large. Note, in particular, that this condition would be satisfied if $\gamma\frac{p(\theta)}{\varepsilon_{q,\theta}} \leqslant 1-\delta$ as σ grows large, since in that case $\overline{J}_u \to 0$ and $\eta \to 0$. □

Proposition 2 (Full)**.** *Under Assumptions 1-5 and if there exists* $0 < \eta < 1 - (1-\delta)^2$ *such that for all* $u \in [\delta(1-s), 1-s]$ *and* $\theta \in [0, \theta_{max}]$*,*

$$\beta \overline{J}_u up(\theta)\, \varepsilon_{p,\theta} \leqslant \eta \frac{\kappa(\theta, u)}{q(\theta)} (\varepsilon_{q,\theta} + \varepsilon_{\kappa,\theta}), \tag{14}$$

where $\overline{J}_u = (1-\beta)^{-1} \left[(1-\gamma) A e^{\overline{z}} \frac{s^{-\frac{\sigma}{\sigma-1}}}{\sigma - 1} + \frac{\beta\gamma\overline{J}(1-\delta)}{(1-\eta)\delta(1-s)} \right]$ *and* $\overline{J} = \frac{1}{1-\beta(1-\delta)} (1-\gamma)\left(Ae^{\overline{z}} - b\right)$*, then there exists a unique equilibrium if*

$$\frac{\beta}{1-\eta} \left| 1 - \delta - \gamma p(\theta) \left(1 + \frac{\varepsilon_{p,\theta}}{\varepsilon_{q,\theta} + \varepsilon_{\kappa,\theta}} \right) \right| < 1. \tag{15}$$

In particular, within the class of mean preserving spreads F_{σ_κ} *with standard deviation* σ_κ *of some distribution* F *with standard deviation 1, i.e., such that* $F_{\sigma_\kappa}(\kappa) = F\left(\mu_\kappa + \sigma_\kappa^{-1}(\kappa - \mu_\kappa)\right)$*, condition* (15) *is satisfied for* σ_κ *large.*

Proof. The general case with heterogeneity and aggregate demand externality results from a direct application of Lemma 1-6. Lemma 2 shows that for $J \in \mathcal{J}$ the free entry problem admits a unique solution. Lemma 4 establishes under condition (14) that the mapping T is a well-defined mapping of $\mathcal{J}$ onto itself. Finally, Lemma 6 demonstrates under condition (15) that the mapping T is a contraction on the Banach space $\mathcal{J}$.

Consider now the limiting case as the dispersion ω of F_ω goes to ∞. In that instance, $\varepsilon_{\kappa,\theta} \to \infty$, implying that

$$\eta = \sup_{\theta,u} \frac{\beta \overline{J}_u up(\theta)\, \varepsilon_{p,\theta}}{\frac{\kappa(\theta,u)}{q(\theta)} (\varepsilon_{q,\theta} + \varepsilon_{\kappa,\theta})} \longrightarrow 0$$

and

$$\frac{\beta}{1-\eta} \left| 1 - \delta - \gamma p(\theta) \left(1 + \frac{\varepsilon_{p,\theta}}{\varepsilon_{q,\theta} + \varepsilon_{\kappa,\theta}} \right) \right| \longrightarrow \beta \left| 1 - \delta - \gamma p(\theta) \right| < 1,$$

which is always satisfied in the limit. We conclude that more heterogeneity diffuses the impact of the complementarity and allows for uniqueness. □

A.6 Auxiliary Lemmas

Lemma 1. *Let* $J \in \mathcal{J}$*. Under Assumption 1-3, for all* $(z,u) \in \Omega$*,* $\theta \in [\Theta(J)](z,u)$*, we have* $\theta \leqslant \theta_{max}$*. As a result,* $\kappa = F^{-1}\left(\frac{\theta u}{M}\right) \leqslant \kappa_{max}$ *where* $\kappa_{max} = min\left[F^{-1}\left(\frac{\theta_{max}(1-s)}{M}\right), \overline{\kappa}\right]$ *and future unemployment* u' *is such that* $\delta(1-s) \leqslant u' \leqslant 1-s$*.*

Proof. Let $(z,u) \in \Omega$ and $\theta \in [\Theta(J)](z,u)$. Since $J \in \mathcal{J}$ is continuous, there must exist at least one solution to the free entry problem ($\theta \in \mathbb{R}_+ \mapsto \Psi(J,z,u,\kappa)$ is either i) greater than 0, in which case $\infty \in [\Theta(J)](z,u)$, ii) less than 0, in which case $0 \in [\Theta(J)](z,u)$, iii) there must exist an interior solution θ such that $\Psi(J,z,u,\theta) = 0$). Under Assumption 3, there can be no solution

$\theta \in [\Theta(J)](z,u)$ with $\theta > \kappa_{max}$ since for $\theta > \kappa_{max}$

$$\Psi(J,z,u,\theta) < \Psi\left(\overline{J},z,u,\theta\right) < \Psi\left(\overline{J},z,u,\theta_{max}\right) < 0,$$

Hence, $\theta \leqslant \kappa_{max}$. As a result, κ is itself bounded above by κ_{max}. Unemployment is bounded below by

$$u' = \delta(1-s) + u(1-\delta-p(\theta)) \geqslant \delta(1-s) + u(1-\delta-p(\theta_{max})) \geqslant \delta(1-s),$$

and bounded above by

$$u' \leqslant \delta(1-s) + (1-s)(1-\delta) = 1-s.$$

□

Lemma 2. *Let $J \in \mathcal{J}$. Under Assumptions 1-4 and if elasticities are such that for all $u \in [\delta(1-s), 1-s]$ and $\theta \in [0,\theta_{max}]$,*

$$\beta\overline{J}_u u p(\theta)\varepsilon_{p,\theta} \leqslant \frac{\kappa(\theta,u)}{q(\theta)}\left[\varepsilon_{q,\theta} + \varepsilon_{\kappa,\theta}\right], \tag{16}$$

then function $\theta \in [0,\theta_{max}] \mapsto \Psi(J,z,u,\theta)$ is strictly decreasing and the free entry problem admits a unique solution, i.e., $[\Theta(J)](z,u)$ is a singleton for all $(z,u) \in \Omega$.

Proof. To show uniqueness of solutions to the free entry problem, we establish that $\Psi(J,z,u,\theta)$ is strictly decreasing under the given parametric conditions. Let $(\theta_1,\theta_2) \in [0,\theta_{max}]^2$ with $\theta_1 < \theta_2$.

$$\begin{aligned}\Psi(J,z,u,\theta_2) - \Psi(J,z,u,\theta_1) &= \beta EJ\left(z', \delta(1-s) + u(1-\delta-p(\theta_2))\right) - \frac{\kappa(\theta_2,u)}{q(\theta_2)} \\ &\quad - \beta EJ\left(z', \delta(1-s) + u(1-\delta-p(\theta_1))\right) + \frac{\kappa(\theta_1,u)}{q(\theta_1)} \\ &= C_0 + C_1,\end{aligned}$$

where

$$\begin{aligned}C_0 &= \beta EJ\left(z', \delta(1-s) + u(1-\delta-p(\theta_2))\right) \\ &\quad - \beta EJ\left(z', \delta(1-s) + u(1-\delta-p(\theta_1))\right) \\ C_1 &= -\frac{\kappa(\theta_2,u)}{q(\theta_2)} + \frac{\kappa(\theta_1,u)}{q(\theta_1)}\end{aligned}$$

Since $J \in \mathcal{J}$ is Lipschitz continuous in u, we have

$$C_0 \leqslant \beta\overline{J}_u u\left[p(\theta_2) - p(\theta_1)\right].$$

Using the Mean Value Theorem, there exists $\tilde{\theta} \in [\theta_1, \theta_2]$ such that

$$\beta \overline{J}_u u \left[p(\theta_2) - p(\theta_1)\right] + C_1 = \left[\beta \overline{J}_u u p'\left(\tilde{\theta}\right) - \frac{d\left(\frac{\kappa}{q}\right)}{d\theta}\left(\tilde{\theta}\right)\right](\theta_2 - \theta_1).$$

We can express the first term as follows

$$\beta \overline{J}_u u p'\left(\tilde{\theta}\right) = \beta \overline{J}_u u p\left(\tilde{\theta}\right) \frac{\varepsilon_{p,\theta}\left(\tilde{\theta}\right)}{\tilde{\theta}},$$

where $\varepsilon_{p,\theta}(\theta) = \frac{d\log p}{d\log\theta}$ is the elasticity of $p(\theta)$ with respect to θ. We express the other term as follows:

$$\begin{aligned}\frac{d\left(\frac{\kappa}{q}\right)}{d\theta}\left(\tilde{\theta}\right) &= \tilde{\kappa}\frac{d\left(\frac{1}{q}\right)}{d\theta}\left(\tilde{\theta}\right) + \frac{1}{q\left(\hat{\theta}\right)}\frac{d\kappa}{d\theta}\left(\tilde{\theta}, u\right) \\ &= -\frac{q'\left(\tilde{\theta}\right)}{q\left(\tilde{\theta}\right)^2}\tilde{\kappa} + \frac{1}{q\left(\tilde{\theta}\right)}\frac{u}{MF'(\tilde{\kappa})} = \frac{\tilde{\kappa}}{q\left(\tilde{\theta}\right)}\frac{1}{\tilde{\theta}}\left[\varepsilon_{q,\theta}\left(\tilde{\theta}\right) + \varepsilon_{\kappa,\theta}\left(\tilde{\theta}, u\right)\right].\end{aligned}$$

Hence, if for all $\theta \in [0, \theta_{max}]$, $u \in [\delta(1-s), 1-s]$, we have

$$\beta \overline{J}_u u p(\theta)\varepsilon_{p,\theta} < \frac{\kappa(\theta, u)}{q(\theta)}\left[\varepsilon_{q,\theta} + \varepsilon_{\kappa,\theta}\right],$$

then function Ψ is strictly decreasing in θ. This guarantees that the free entry problem admits exactly one solution. □

Lemma 3. *Under the same conditions as Lemma 2, the solution to the free entry problem satisfies:*

1. *For* $J \in \mathcal{J}$, $z \in [\underline{z}, \overline{z}]$, $(u_1, u_2) \in [\delta(1-s), 1-s]^2$, $u_1 < u_2$, *and* $\theta_i \in [\Theta(J)](z, u_i)$, $i = 1, 2$, *then we have*

$$\max_{\substack{u \in [u_1, u_2], \\ \theta \in [\theta_2, \theta_1]}} \frac{\left[\beta(1 - \delta - p(\theta))\overline{J}_u + \frac{\kappa(\theta,u)}{uq(\theta)}\varepsilon_{\kappa,u}\right](u_2 - u_1)}{\frac{1}{\theta}\left[\frac{\kappa(\theta,u)}{q(\theta)}(\varepsilon_{q,\theta} + \varepsilon_{\kappa,\theta}) - up(\theta)\varepsilon_{p,\theta}\beta\overline{J}_u\right]} \leqslant \theta_2 - \theta_1 < 0.$$

2. *For* $(J_1, J_2) \in \mathcal{J}^2, J_1 \leqslant J_2, (z, u) \in \Omega$ *and* $\kappa_i \in [K(J_i)](z, u)$, $i = 1, 2$, *then we have*

$$0 \leqslant \theta_2 - \theta_1 \leqslant \frac{\beta \|J_2 - J_1\|}{\min_{\theta \in [\theta_1, \theta_2]} \frac{1}{\theta}\left[\frac{\kappa(\theta,u)}{q(\theta)}(\varepsilon_{q,\theta} + \varepsilon_{\kappa,\theta}) - \beta\overline{J}_u u p(\theta)\varepsilon_{p,\theta}\right]}.$$

Proof. 1. For $\lambda \in [0,1]$, let $u_\lambda = \lambda u_2 + (1-\lambda) u_1$ and denote $(\theta_\lambda, \kappa_\lambda)$ the corresponding market tightness and marginal cost of entry. Since $J \in \mathcal{J}$ is Lipschitz continuous in u, $\Psi(J,z,u,\theta)$ is Lipschitz continuous in u and θ. The function $\Psi(J,z,u_\lambda,\theta_\lambda)$ is also Lipschitz continuous in λ. Hence, it is also absolutely continuous and differentiable almost everywhere. The Implicit Function Theorem tells us that θ_λ is also absolutely continuous, hence differentiable almost everywhere in λ. In particular, we can write the following:

$$\theta_2 - \theta_1 = \int_0^1 \frac{d\theta_\lambda}{d\lambda} d\lambda.$$

For all $\lambda \in [0,1]$ where J is differentiable in u, we compute

$$\frac{\partial}{\partial \lambda}\Psi(J,z,u_\lambda,\theta) = \left[\beta(1-\delta-p(\theta)) EJ_u\left(z',u'_\lambda\right) - \frac{\kappa(\theta,u_\lambda)}{u_\lambda q(\theta)}\varepsilon_{\kappa,u}(\theta,u_\lambda)\right](u_2-u_1),$$

and repeating the same calculations as in Lemma 2, we have

$$\frac{\partial}{\partial \theta}\Psi(J,z,u_\lambda,\theta) = -\frac{1}{\theta}\left[\frac{\kappa(\theta,u_\lambda)}{q(\theta)}(\varepsilon_{q,\theta}+\varepsilon_{\kappa,\theta}) - u_\lambda p(\theta)\varepsilon_{p,\theta}\beta EJ_u\left(z',u'_\lambda\right)\right].$$

The Implicit Function Theorem tells us that

$$\frac{d\theta_\lambda}{d\lambda} = -\frac{\frac{\partial}{\partial\lambda}\Psi(J,z,u_\lambda,\theta)}{\frac{\partial}{\partial\theta}\Psi(J,z,u_\lambda,\theta)} = \frac{\left[\beta(1-\delta-p(\theta_\lambda))EJ_u(z',u'_\lambda) - \frac{\kappa(\theta_\lambda,u_\lambda)}{u_\lambda q(\theta_\lambda)}\varepsilon_{\kappa,u}\right](u_2-u_1)}{\frac{1}{\theta}\left[\frac{\kappa(\theta_\lambda,u_\lambda)}{q(\theta_\lambda)}(\varepsilon_{q,\theta}+\varepsilon_{\kappa,\theta}) - u_\lambda p(\theta_\lambda)\varepsilon_{p,\theta}\beta EJ_u\left(z',u'_\lambda\right)\right]}.$$

Hence,

$$\begin{aligned}|\theta_2-\theta_1| &\leqslant \int_0^1 \left|\frac{d\theta_\lambda}{d\lambda}\right| d\lambda \\ &\leqslant \max_{\substack{u\in[u_1,u_2],\\ \theta\in[\theta_2,\theta_1]}} \frac{\left[\beta(1-\delta-p(\theta))\overline{J}_u + \frac{\kappa(\theta,u)}{uq(\theta)}\varepsilon_{\kappa,u}\right](u_2-u_1)}{\frac{1}{\theta}\left[\frac{\kappa(\theta,u)}{q(\theta)}(\varepsilon_{q,\theta}+\varepsilon_{\kappa,\theta}) - up(\theta)\varepsilon_{p,\theta}\beta\overline{J}_u\right]}.\end{aligned}$$

2. Evaluate $\Psi(J_2,z,u,\theta_1)$:

$$\begin{aligned}\Psi(J_2,z,u,\theta_1) &= \beta EJ_2\left(z',\delta(1-s)+u(1-\delta-p(\theta_1))\right) - \frac{\kappa(\theta_1,u)}{q(\theta_1)} \\ &\geqslant \beta EJ_1\left(z',\delta(1-s)+u(1-\delta-p(\theta_1))\right) - \frac{\kappa(\theta_1,u)}{q(\theta_1)}.\end{aligned}$$

Since Ψ is decreasing in θ, this implies that $\theta_2 \geqslant \theta_1$. To derive the upper bound on $\theta_2-\theta_1$, we compute

$$|\Psi(J_2,z,u,\theta) - \Psi(J_1,z,u,\theta)| \leqslant \beta\|J_2-J_1\|,$$

and use the Implicit Function Theorem for Lipschitz continuous functions, which tells us that

$$|\theta_2 - \theta_1| \leqslant \frac{\beta \|J_2 - J_1\|}{\min_{\theta \in [\theta_2, \theta_1]} \frac{1}{\theta} \left[\frac{\kappa(\theta,u)}{q(\theta)} \left(\varepsilon_{q,\theta} + \varepsilon_{\kappa,\theta} \right) - up\left(\theta\right) \varepsilon_{p,\theta} \beta E J_u \left(z', u'\right) \right]}.$$

□

Lemma 4. *Under Assumptions 1-5, if there exists* $0 < \eta < 1 - (1-\delta)^2$ *such that* $\forall (z,u) \in \Omega, \theta(z,u) \in [\Theta(J)](z,u)$,

$$\beta \overline{J}_u up(\theta) \varepsilon_{p,\theta} \leqslant \eta \frac{\kappa(\theta, u)}{q(\theta)} \left[\varepsilon_{q,\theta} + \varepsilon_{\kappa,\theta} \right], \tag{17}$$

then the mapping T *is a well-defined self-map on* $\mathcal{J}$.

Proof. Let $J \in \mathcal{J}$. We verify each property individually.

1. *$J(z,u)$ belongs to $\left[\underline{J}, \overline{J}\right]$ for $(z,u) \in \Omega$.* Recall the definition,

$$[T(J)](z,u) = (1-\gamma)\left(Ae^{z}(1-u)^{\frac{1}{\sigma-1}} - b\right) + \beta(1-\delta-\gamma p(\theta)) E\left[J\left(z',u'\right)|z\right].$$

Hence,

$$[T(J)](z,u) \leqslant (1-\gamma)\left(Ae^{\overline{z}} - b\right) + \beta(1-\delta)\overline{J},$$

which is guaranteed since $\overline{J} = \frac{1}{1-\beta(1-\delta)}(1-\gamma)\left(Ae^{\overline{z}} - b\right)$. Similarly,

$$[T(J)](z,u) \geqslant (1-\gamma)\left(Ae^{\underline{z}} s^{\frac{1}{\sigma-1}} - b\right) + \beta(1-\delta-\gamma)\underline{J},$$

which is satisfied since $\underline{J} = \frac{1}{1-\beta(1-\delta-\gamma)}(1-\gamma)\left(Ae^{\underline{z}} s^{\frac{1}{\sigma-1}} - b\right)$, which is strictly greater than 0 under Assumption 5.

2. *J is continuous in z.* Let $(z,u) \in \Omega$. According to Lemma 2, there exists a unique solution $\hat{\kappa}(z,u)$ to the free entry problem. In particular, since $J \in \mathcal{J}$, function $\Psi(J,z,u,\kappa)$ is continuous in z and κ. The implicit function theorem tells us that $\hat{\kappa}(z,u)$ is continuous in z. Being a algebraic combination of continuous functions, $T(J)$ is thus continuous in z.

3. *J is decreasing and Lipschitz continuous in u of modulus $\overline{J}_u$.* Let $(u_1,u_2) \in [\delta(1-s), 1-s]$ with $u_1 < u_2$. For $\lambda \in [0,1]$, denote $u_\lambda = \lambda u_2 + (1-\lambda)u_1$ and denote $(\theta_\lambda, \kappa_\lambda)$ the associated market tightnesses and cutoffs. Since $J \in \mathcal{J}$, it is Lipschitz continuous in u and so is $T(J)$. Hence, they are absolutely continuous and differentiable almost everywhere. In particular, we can write the following

$$[T(J)](z,u_2) - [T(J)](z,u_1) = \int_0^1 \frac{d}{d\lambda}[T(J)](z,u_\lambda)\, d\lambda.$$

Take $\lambda \in [0,1]$ such that $[T(J)](z,u_\lambda)$ and $EJ(z',u'_\lambda)$ are differentiable in λ and compute

$$\frac{d}{d\lambda}[T(J)](z,u_\lambda) = C_0 + \beta(C_1 + C_2 + C_3)$$

where

$$C_0 = -(1-\gamma)Ae^z\frac{1}{\sigma-1}(1-u_\lambda)^{-\frac{\sigma}{\sigma-1}}(u_2-u_1)$$
$$C_1 = (1-\delta-\gamma p(\theta_\lambda))(1-\delta-p(\theta_\lambda))EJ_u\left(z',\delta(1-s)+u_\lambda(1-\delta-p(\theta_\lambda))\right)(u_2-u_1)$$
$$C_2 = -(1-\delta-\gamma p(\theta_\lambda))u_\lambda p'(\theta_\lambda)\frac{d\theta_\lambda}{d\lambda}EJ_u\left(z',u'_\lambda\right)$$
$$C_3 = -\gamma p'(\theta_\lambda)\frac{d\theta_\lambda}{d\lambda}EJ\left(z',u'_\lambda\right).$$

Using the expression for $\frac{d\theta_\lambda}{d\lambda}$ from Lemma 3, we have the following:

$$\begin{aligned}
p'(\theta_\lambda)\left|\frac{d\theta_\lambda}{d\lambda}\right| &= p(\theta_\lambda)\varepsilon_{p,\theta}(\theta_\lambda)\frac{-\beta(1-\delta-p(\theta_\lambda))EJ_u(z',u'_\lambda)+\frac{\kappa(\theta_\lambda,u_\lambda)}{uq(\theta_\lambda)}\varepsilon_{\kappa,u}}{\frac{\kappa(\theta_\lambda,u_\lambda)}{q(\theta_\lambda)}(\varepsilon_{q,\theta}+\varepsilon_{\kappa,\theta})-u_\lambda p(\theta_\lambda)\varepsilon_{p,\theta}\beta EJ_u\left(z',u'_\lambda\right)}(u_2-u_1) \\
&\leqslant p(\theta_\lambda)\varepsilon_{p,\theta}\frac{\beta(1-\delta-p(\theta_\lambda))\overline{J}_u+\frac{\kappa(\theta_\lambda,u_\lambda)}{u_\lambda q(\theta_\lambda)}\varepsilon_{\kappa,u}}{\frac{\kappa(\theta_\lambda,u_\lambda)}{q(\theta_\lambda)}(\varepsilon_{q,\theta}+\varepsilon_{\kappa,\theta})-u_\lambda p(\theta_\lambda)\varepsilon_{p,\theta}\beta\overline{J}_u}(u_2-u_1).
\end{aligned}$$

We now compute $C_1 + C_2$:

$$\begin{aligned}
|C_1+C_2| &= (1-\delta-\gamma p(\theta_\lambda))\overline{J}_u\left((1-\delta-p(\theta_\lambda))(u_2-u_1)+up'(\theta_\lambda)\left|\frac{d\theta}{d\lambda}\right|\right) \\
&= (1-\delta-\gamma p(\theta_\lambda))\left(1-\delta-p(\theta_\lambda)+up(\theta_\lambda)\varepsilon_{p,\theta}\frac{\beta(1-\delta-p(\theta_\lambda))\overline{J}_u+\frac{\kappa(\theta_\lambda,u_\lambda)}{u_\lambda q(\theta_\lambda)}\varepsilon_{\kappa,u}}{\frac{\kappa(\theta_\lambda,u_\lambda)}{q(\theta_\lambda)}(\varepsilon_{q,\theta}+\varepsilon_{\kappa,\theta})-u_\lambda p(\theta_\lambda)\varepsilon_{p,\theta}\beta\overline{J}_u}\right) \\
&\times \overline{J}_u(u_2-u_1) \\
&= (1-\delta-\gamma p(\theta_\lambda))\left[(1-\delta-p(\theta_\lambda))\left(1+\frac{up(\theta_\lambda)\varepsilon_{p,\theta}\beta\overline{J}_u}{\frac{\kappa(\theta_\lambda,u_\lambda)}{q(\theta_\lambda)}(\varepsilon_{q,\theta}+\varepsilon_{\kappa,\theta})-u_\lambda p(\theta_\lambda)\varepsilon_{p,\theta}\beta\overline{J}_u}\right)\right. \\
&\left.+\frac{\frac{\kappa(\theta_\lambda,u_\lambda)}{q(\theta_\lambda)}p(\theta_\lambda)\varepsilon_{p,\theta}\varepsilon_{\kappa,\theta}}{\frac{\kappa(\theta_\lambda,u_\lambda)}{q(\theta_\lambda)}(\varepsilon_{q,\theta}+\varepsilon_{\kappa,\theta})-u_\lambda p(\theta_\lambda)\varepsilon_{p,\theta}\beta\overline{J}_u}\right]\overline{J}_u(u_2-u_1),
\end{aligned}$$

which, under condition (17), yields

$$|C_1+C_2| \leqslant \overline{J}_u(1-\delta-\gamma p(\theta_\lambda))\left[\frac{1-\delta-p(\theta_\lambda)}{1-\eta}+\frac{p(\theta_\lambda)\varepsilon_{p,\theta}\varepsilon_{\kappa,\theta}}{(1-\eta)(\varepsilon_{q,\theta}+\varepsilon_{\kappa,\theta})}\right](u_2-u_1).$$

Note that since $\varepsilon_{p,\theta} = 1 + \varepsilon_{q,\theta}$ and $\varepsilon_{q,\theta} < 0$, we have $\frac{\varepsilon_{p,\theta}\varepsilon_{\kappa,\theta}}{-\varepsilon_{q,\theta}+\varepsilon_{\kappa,\theta}} \leqslant \frac{\varepsilon_{\kappa,\theta}}{-\varepsilon_{q,\theta}+\varepsilon_{\kappa,\theta}} \leq 1$. Hence,

$$|C_1 + C_2| \leq \overline{J}_u \left(1 - \delta - \gamma p\left(\theta_\lambda\right)\right) \frac{1-\delta}{1-\eta} \left(u_2 - u_1\right) \leq \overline{J}_u \left(u_2 - u_1\right).$$

Similarly, we can simplify C_3 as follows:

$$\begin{aligned}
|C_3| &= -\gamma p'\left(\theta_\lambda\right) \frac{d\theta_\lambda}{d\lambda} EJ\left(z', u'_\lambda\right) \leq \gamma p\left(\theta_\lambda\right) \varepsilon_{p,\theta} \frac{\beta\left(1 - \delta - p\left(\theta_\lambda\right)\right) \overline{J}_u + \frac{\kappa(\theta_\lambda, u_\lambda)}{u_\lambda q(\theta_\lambda)} \varepsilon_{\kappa,u}}{\frac{\kappa(\theta_\lambda, u_\lambda)}{q(\theta_\lambda)} \left(\varepsilon_{q,\theta} + \varepsilon_{\kappa,\theta}\right) - u_\lambda p\left(\theta_\lambda\right) \varepsilon_{p,\theta} \beta \overline{J}_u} \left(u_2 - u_1\right) \overline{J} \\
&\leqslant \frac{\gamma \overline{J}}{u_\lambda \left(1 - \eta\right)} \left[\eta\left(1 - \delta\right) + \left(1 - \eta\right) p\left(\theta_\lambda\right) \frac{\varepsilon_{\kappa,\theta}}{\varepsilon_{q,\theta} + \varepsilon_{\kappa,\theta}}\right] \left(u_2 - u_1\right) \\
&\leqslant \frac{\gamma \overline{J}\left(1 - \delta\right)}{\left(1 - \eta\right) u_\lambda} \left[\eta + \left(1 - \eta\right) \frac{\varepsilon_{\kappa,\theta}}{\varepsilon_{q,\theta} + \varepsilon_{\kappa,\theta}}\right] \left(u_2 - u_1\right).
\end{aligned}$$

We conclude that $T(J)$ is Lipschitz of modulus $\overline{J}_u$ if

$$\begin{aligned}
|C_0 + \beta\left(C_1 + C_2 + C_3\right)| = \left(1 - \gamma\right) A e^{\overline{z}} \frac{s^{-\frac{\sigma}{\sigma-1}}}{\sigma - 1} + \frac{\beta \gamma \overline{J}\left(1 - \delta\right)}{\left(1 - \eta\right) \delta\left(1 - s\right)} \left[\eta + \left(1 - \eta\right) \frac{\varepsilon_{\kappa,\theta}}{\varepsilon_{q,\theta} + \varepsilon_{\kappa,\theta}}\right] \\
+ \beta \overline{J}_u \left(u_2 - u_1\right) \leqslant \overline{J}_u \left(u_2 - u_1\right),
\end{aligned}$$

which is clearly satisfied since $\overline{J}_u = \left(1 - \beta\right)^{-1} \left[\left(1 - \gamma\right) A e^{\overline{z}} \frac{s^{-\frac{\sigma}{\sigma-1}}}{\sigma-1} + \frac{\beta\gamma\overline{J}(1-\delta)}{(1-\eta)\delta(1-s)} \left(\eta + \left(1 - \eta\right) \frac{\overline{\varepsilon}_{\kappa,\theta}}{\underline{\varepsilon}_{q,\theta} + \underline{\varepsilon}_{\kappa,\theta}}\right)\right]$.
We conclude that $T(J)$ belongs to $\mathcal{J}$ and T is a well-defined self-map. ☐

Lemma 5. *Under the same conditions as Lemma 4 and for parameters such that* $\forall (z, u) \in \Omega$, $\theta(z, u) \in [\Theta(J)][z, u]$ *, we have*

$$1 - \delta \geq \gamma p(\theta) \left[1 + \frac{1}{1 - 2\eta} \frac{\varepsilon_{p,\theta}}{\varepsilon_{q,\theta} + \varepsilon_{\kappa,\theta}}\right], \tag{18}$$

then the mapping T is monotone, i.e., if $(J_1, J_2) \in \mathcal{J}^2$, $J_1 \leq J_2$, then $T(J_1) \leq T(J_2)$.

Proof. Let $(J_1, J_2) \in \mathcal{J}^2$, $J_1 \leq J_2$. For $\lambda \in [0, 1]$, denote $J_\lambda = \lambda J_2 + (1 - \lambda) J_1$ and denote $(\theta_\lambda, \kappa_\lambda)$ the associated market tightnesses and cutoffs. Since $J_i \in \mathcal{J}, i = 1, 2$, it is Lipschitz continuous in u and so is $T(J)$. They are also Lipschitz continuous in λ. Hence, they are absolutely continuous and differentiable almost everywhere. In particular, we can write the following

$$[T(J_2)](z, u) - [T(J_1)](z, u) = \int_0^1 \frac{d}{d\lambda} [T(J_\lambda)](z, u_\lambda) \, d\lambda.$$

We now evaluate $\frac{d}{d\lambda}\left[T\left(J_\lambda\right)\right]$:

$$\frac{d}{d\lambda}\left[T\left(J_\lambda\right)\right](z,u) = \beta\left(1-\delta-\gamma p\left(\theta_\lambda\right)\right)\beta E\left[J_2\left(z',u'_\lambda\right)-J_1\left(z',u'_\lambda\right)\right] \\ -\beta p'\left(\theta_\lambda\right)\frac{d\theta_\lambda}{d\lambda}\left\{\gamma E\left[J_\lambda\left(z',u'_\lambda\right)\right]+\left(1-\delta-\gamma p\left(\theta_\lambda\right)\right)u_\lambda E J_{\lambda,u}\left(z',u'_\lambda\right)\right\}.$$

Take $\lambda\in[0,1]$ such that $\left[T\left(J_\lambda\right)\right](z,u)$ and $EJ\left(z',u'_\lambda\right)$ are differentiable in λ. Using the expression for $\frac{d\theta_\lambda}{d\lambda}$ from Lemma 3, we have

$$p'\left(\theta_\lambda\right)\frac{d\theta_\lambda}{d\lambda} = p\left(\theta_\lambda\right)\varepsilon_{p,\theta}\frac{\beta E\left[J_2\left(z',u'_\lambda\right)-J_1\left(z',u'_\lambda\right)\right]}{\frac{\kappa(\theta,u)}{q(\theta)}\left(\varepsilon_{q,\theta}+\varepsilon_{\kappa,\theta}\right)-up\left(\theta\right)\varepsilon_{p,\theta}\beta EJ_u\left(z',u'\right)} \\ \leqslant p\left(\theta_\lambda\right)\varepsilon_{p,\theta}\frac{\beta E\left[J_2\left(z',u'_\lambda\right)-J_1\left(z',u'_\lambda\right)\right]}{\left(1-\eta\right)\frac{\kappa(\theta,u)}{q(\theta)}\left(\varepsilon_{q,\theta}+\varepsilon_{\kappa,\theta}\right)}.$$

Hence, we have

$$\frac{d}{d\lambda}\left[T\left(J_\lambda\right)\right](z,u) = \beta E\left[J_2\left(z',u'_\lambda\right)-J_1\left(z',u'_\lambda\right)\right]\Bigg[1-\delta-\gamma p\left(\theta_\lambda\right) \\ -\frac{\beta p\left(\theta_\lambda\right)\varepsilon_{p,\theta}}{\left(1-\eta\right)\frac{\kappa(\theta,u)}{q(\theta)}\left(-\varepsilon_{q,\theta}+\varepsilon_{\kappa,\theta}\right)}\left(\gamma E\left[J_\lambda\left(z',u'_\lambda\right)\right]+\left(1-\delta-\gamma p\left(\theta_\lambda\right)\right)u_\lambda EJ_{\lambda,u}\left(z',u'_\lambda\right)\right)\Bigg] \\ \geqslant \beta E\left[J_2\left(z',u'_\lambda\right)-J_1\left(z',u'_\lambda\right)\right]\left[1-\delta-\gamma p\left(\theta_\lambda\right)-\frac{\gamma p\left(\theta_\lambda\right)\varepsilon_{p,\theta}}{\left(1-\eta\right)\left(\varepsilon_{q,\theta}+\varepsilon_{\kappa,\theta}\right)}-\frac{\eta\left(1-\delta-\gamma p\left(\theta_\lambda\right)\right)}{1-\eta}\right] \\ \geqslant \beta E\left[J_2\left(z',u'_\lambda\right)-J_1\left(z',u'_\lambda\right)\right]\left[\left(1-\frac{\eta}{1-\eta}\right)\left(1-\delta-\gamma p\left(\theta_\lambda\right)\right)-\frac{\gamma p\left(\theta_\lambda\right)\varepsilon_{p,\theta}}{\left(1-\eta\right)\left(\varepsilon_{q,\theta}+\varepsilon_{\kappa,\theta}\right)}\right],$$

where we have used condition (17) and the fact that $p\left(\theta_\lambda\right)\leqslant 1-\delta$ and $\frac{\kappa_\lambda}{q\left(\theta_\lambda\right)}\geqslant E\left[J_\lambda\right]$ for $\theta_\lambda\leqslant\theta_{max}$. Hence, the mapping is monotone if

$$\left(1-\frac{\eta}{1-\eta}\right)\left(1-\delta-\gamma p\left(\theta_\lambda\right)\right)\geqslant p\left(\theta_\lambda\right)\frac{\gamma}{1-\eta}\frac{\varepsilon_{p,\theta}}{\varepsilon_{q,\theta}+\varepsilon_{\kappa,\theta}},$$

which can be simplified to equation (18). □

Lemma 6. *Under the same conditions as Lemma 4 and for parameters such that* $\forall\left(z,u\right)\in\Omega$, $\theta\left(z,u\right)\in\left[\Theta\left(J\right)\right]\left[z,u\right]$ *, we have*

$$\frac{\beta}{1-\eta}\left|1-\delta-\gamma p\left(\theta\right)\left(1+\frac{\varepsilon_{p,\theta}}{\varepsilon_{q,\theta}+\varepsilon_{\kappa,\theta}}\right)\right|<1,$$

then the mapping T is a contraction.

Proof. This proof repeats several steps of Lemma 5. Let $\left(J_1,J_2\right)\in\mathcal{J}^2$. For $\lambda\in[0,1]$, denote $J_\lambda=\lambda J_2+\left(1-\lambda\right)J_1$ and denote $\left(\theta_\lambda,\kappa_\lambda\right)$ the associated market tightnesses and cutoffs. Since $J_i\in\mathcal{J}$,$i=1,2$, it is Lipschitz continuous in u and so is $T\left(J\right)$. They are also Lipschitz continuous in λ. Hence, they are absolutely continuous and differentiable almost everywhere. In particular, we can write the following

$$[T(J_2)](z,u) - [T(J_1)](z,u) = \int_0^1 \frac{d}{d\lambda}[T(J_\lambda)](z,u_\lambda)\,d\lambda.$$

We now evaluate $\frac{d}{d\lambda}[T(J_\lambda)]$:

$$\begin{aligned}\frac{d}{d\lambda}[T(J_\lambda)](z,u) &= \beta(1-\delta-\gamma p(\theta_\lambda))\beta E\left[J_2\left(z',u'_\lambda\right) - J_1\left(z',u'_\lambda\right)\right] \\ &\quad - \beta p'(\theta_\lambda)\frac{d\theta_\lambda}{d\lambda}\left\{\gamma E\left[J_\lambda\left(z',u'_\lambda\right)\right] + (1-\delta-\gamma p(\theta_\lambda))u_\lambda E J_{\lambda,u}\left(z',u'_\lambda\right)\right\}.\end{aligned}$$

Take $\lambda \in [0,1]$ such that $[T(J_\lambda)](z,u)$ and $EJ(z',u'_\lambda)$ are differentiable in λ. Using the same expressions as in Lemma 5, we have

$$\begin{aligned}\frac{d}{d\lambda}[T(J_\lambda)](z,u) &= \beta E\left[J_2\left(z',u'_\lambda\right) - J_1\left(z',u'_\lambda\right)\right]\Bigg[1-\delta-\gamma p(\theta_\lambda) \\ &\quad - \frac{\beta p(\theta_\lambda)\varepsilon_{p,\theta}}{(1-\eta)\frac{\kappa(\theta,u)}{q(\theta)}(\varepsilon_{q,\theta}+\varepsilon_{\kappa,\theta})}\left(\gamma E\left[J_\lambda\left(z',u'_\lambda\right)\right] + (1-\delta-\gamma p(\theta_\lambda))u_\lambda E J_{\lambda,u}\left(z',u'_\lambda\right)\right)\Bigg] \\ &\leqslant \beta\|J_2-J_1\|\left[1-\delta-\gamma p(\theta_\lambda) - \frac{\gamma p(\theta_\lambda)\varepsilon_{p,\theta}}{(1-\eta)(\varepsilon_{q,\theta}+\varepsilon_{\kappa,\theta})} + \frac{\eta(1-\delta-\gamma p(\theta_\lambda))}{1-\eta}\right] \\ &\leqslant \beta\|J_2-J_1\|\left[\frac{1}{1-\eta}(1-\delta-\gamma p(\theta_\lambda)) - \frac{\gamma p(\theta_\lambda)\varepsilon_{p,\theta}}{(1-\eta)(\varepsilon_{q,\theta}+\varepsilon_{\kappa,\theta})}\right].\end{aligned}$$

Hence, the mapping is a contraction if

$$\frac{\beta}{1-\eta}\left|1-\delta-\gamma p(\theta_\lambda)\left(1+\frac{\varepsilon_{p,\theta}}{\varepsilon_{q,\theta}+\varepsilon_{\kappa,\theta}}\right)\right| < 1.$$

□