Supplement to “Endogenous Production Networks under Supply Chain Uncertainty.”

Alexandr Kopytov
University of Rochester

Bineet Mishra
Cornell University

Kristoffer Nimark
Cornell University

Mathieu Taschereau-Dumouchel
Cornell University

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Abstract

This is the Supplemental Appendix for “Endogenous Production Networks under Supply Chain Uncertainty”.

JEL Classifications: E32, C67, D57, D80, D85

A Additional derivations and results

This appendix contains additional derivations that are used in the main text.

A.1 Derivation of the stochastic discount factor

The Lagrange multiplier on the budget constraint of the household captures the value of an extra unit of the numeraire and serves as stochastic discount factor for firms to compare profits across states of the world. The following lemma shows how to derive the expression in the main text.

Lemma 10. *The Lagrange multiplier on the budget constraint of the household* (4) *is*

$$\Lambda = \frac{u'(Y)}{\overline{P}},$$

where $Y = \prod_{i=1}^{n} \left(\beta_i^{-1} C_i\right)^{\beta_i}$ *and* $\overline{P} = \prod_{i=1}^{n} P_i^{\beta_i}$.

Proof. The household makes decisions after the realization of the state of the world ε. The state-specific maximization problem has a concave objective function and a convex constraint set so that first-order conditions are sufficient to characterize optimal decisions. The Lagrangian is

$$u\left(\left(\frac{C_1}{\beta_1}\right)^{\beta_1} \times \cdots \times \left(\frac{C_n}{\beta_n}\right)^{\beta_n}\right) - \Lambda\left(\sum_{i=1}^{n} P_i C_i - 1\right)$$

and the first-order condition with respect to C_i is

$$\beta_i u'(Y) Y = \Lambda P_i C_i. \tag{A.1}$$

Summing over i on both sides and using the binding budget constraint yields

$$u'(Y) Y = \Lambda, \tag{A.2}$$

which, together with (A.1), implies that

$$P_i C_i = \beta_i. \tag{A.3}$$

We can also plug back the first-order condition in $Y = \prod_{i=1}^{n} \left(\beta_i^{-1} C_i\right)^{\beta_i}$ to find

$$Y = \prod_{i=1}^{n} \left(\beta_i^{-1} C_i\right)^{\beta_i} = \prod_{i=1}^{n} \left(\beta_i^{-1} \frac{\beta_i u'(Y) Y}{\Lambda P_i}\right)^{\beta_i}$$

$$\Lambda = u'(Y) \prod_{i=1}^{n} P_i^{-\beta_i} \tag{A.4}$$

which, combined with (A.2), yields

$$Y = \prod_{i=1}^{n} P_i^{-\beta_i}. \tag{A.5}$$

This last equation implicitly defines a price index $\overline{P} = \prod_{i=1}^{n} P_i^{\beta_i}$ such that $\overline{P} Y = 1$. Combining that last equation with (A.2) yields the result. □

A.2 Derivation of the unit cost function

The cost minimization problem of the firm is

$$K_i(\alpha_i, P) = \min_{L_i, X_i} \left(L_i + \sum_{j=1}^{n} P_j X_{ij}\right)$$
$$\text{subject to } F(\alpha_i, L_i, X_i) \geq 1,$$

where F is given by (1). The first-order conditions are

$$L_i = \theta \left(1 - \sum_{j=1}^{n} \alpha_{ij}\right) F\left(\alpha_i, L_i, X_i\right),$$
$$P_j X_{ij} = \theta \alpha_{ij} F\left(\alpha_i, L_i, X_i\right),$$

where θ is the Lagrange multiplier. Plugging these expressions back into the objective function, we see that $K_i\left(\alpha_i, P\right) = \theta$ since $F\left(\alpha_i, L_i, X_i\right) = 1$ at the optimum. Now, plugging the first-order conditions in the production function we find

$$1 = e^{\varepsilon_i} A_i\left(\alpha_i\right) \theta \prod_{j=1}^{n} P_j^{-\alpha_{ij}},$$

which is the result.

A.3 Derivation of the first-order condition of the firm (19)

At an interior solution, the first-order conditions associated with problem (17) are

$$0 = -\frac{da_i\left(\alpha_i\right)}{d\alpha_{ij}} + \mathcal{R}_j.$$

We can write these equations in vector form as $f\left(\alpha_i, \mathcal{R}\right) = 0$, with element j given by

$$f_j\left(\alpha_i, \mathcal{R}\right) = -\frac{da_i\left(\alpha_i\right)}{d\alpha_{ij}} + \mathcal{R}_j.$$

A solution α_i to the first-order conditions corresponds to $f\left(\alpha_i, \mathcal{R}\right) = 0$. The Jacobian of f with respect to α_i is $-H_i$ where H_i is the Hessian of a_i. The Jacobian of f with respect to the vector $\mathcal{R}$ is the identity matrix. From the implicit function theorem, we can therefore write

$$\frac{\partial \alpha_{ij}}{\partial \mathcal{R}_k} = \left[\frac{\partial \alpha_i}{\partial \mathcal{R}}\right]_{jk} = -\left[\left[\frac{\partial f}{\partial \alpha_i}\right]^{-1} \times \frac{\partial f}{\partial \mathcal{R}}\right]_{jk} = \left[H_i^{-1}\right]_{jk}.$$

A.4 Derivation of $\bar{a}$ and $\alpha\left(\omega\right)$ under quadratic TFP shifter

In this appendix, we solve the problem (23) at an interior solution when the TFP shifter functions $\left(a_1, \ldots, a_n\right)$ are of the form (2). From (23), the planner seeks to maximize

$$\sum_i \omega_i \left(\frac{1}{2} \alpha_i^\top H_i \alpha_i - \left(\alpha_i^\circ\right)^\top H \alpha_i\right),$$

subject to $\omega^\top = \beta^\top \mathcal{L}(\alpha)$ or, since $I - \alpha$ is always invertible for $\alpha \in \mathcal{A}$, $\alpha^\top \omega = \omega - \beta$. We can rewrite this problem as a standard quadratic program

$$\min_{\alpha \in \mathcal{A}} \frac{1}{2} x^\top Q x + c^\top x,$$

subject to $Ex = d$, where

$$\underbrace{Q}_{n^2 \times n^2} = - \begin{bmatrix} \omega_1 H_1 & & 0 \\ & \ddots & \\ 0 & & \omega_n H_n \end{bmatrix},$$

and where the constraint

$$\alpha^\top \omega = \omega - \beta,$$

becomes

$$\underbrace{\underbrace{[I\omega_1, \ldots, I\omega_n]}}_{\substack{E \\ n \times n^2}} \underbrace{\begin{bmatrix} \alpha_{11} \\ \vdots \\ \alpha_{1n} \\ \vdots \\ \alpha_{n1} \\ \vdots \\ \alpha_{nn} \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} \omega_1 - \beta_1 \\ \vdots \\ \omega_n - \beta_n \end{bmatrix}}_{d},$$

and where $c^\top = \begin{bmatrix} \omega_1 (\alpha_1^\circ)^\top H_1 & \ldots & \omega_n (\alpha_n^\circ)^\top H_n \end{bmatrix}$. The solution to this problem is well-known and given by

$$\begin{bmatrix} Q & E^\top \\ E & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} -c \\ d \end{bmatrix}.$$

Proof. Using the block matrix inverse equation we can write

$$\begin{bmatrix} Q & E^\top \\ E & 0 \end{bmatrix}^{-1} = \begin{bmatrix} Q^{-1} - Q^{-1}E^\top \left(EQ^{-1}E^\top\right)^{-1} EQ^{-1} & Q^{-1}E^\top \left(EQ^{-1}E^\top\right)^{-1} \\ \left(EQ^{-1}E^\top\right)^{-1} EQ^{-1} & -\left(EQ^{-1}E^\top\right)^{-1} \end{bmatrix},$$

such that the solution to the optimization problem is

$$x = Q^{-1}E^\top \left(EQ^{-1}E^\top\right)^{-1} d - \left(Q^{-1} - Q^{-1}E^\top \left(EQ^{-1}E^\top\right)^{-1} EQ^{-1}\right) c.$$

Simple matrix algebra implies that

$$EQ^{-1} = - \begin{bmatrix} H_1^{-1} & \ldots & H_n^{-1} \end{bmatrix},$$

and

$$EQ^{-1}E^{\top} = -\sum_i \omega_i H_i^{-1} = -D,$$

where we explicitly define the square matrix D.

It follows that

$$x = \begin{bmatrix} H_1^{-1} \\ \vdots \\ H_n^{-1} \end{bmatrix} D^{-1} \begin{bmatrix} \omega_1 - \beta_1 \\ \vdots \\ \omega_n - \beta_n \end{bmatrix} + \begin{bmatrix} \omega_1^{-1} H_1^{-1} & & 0 \\ & \ddots & \\ 0 & & \omega_n^{-1} H_n^{-1} \end{bmatrix} \begin{bmatrix} \omega_1 H_1 \alpha_1^\circ \\ \vdots \\ \omega_n H_n \alpha_n^\circ \end{bmatrix}$$
$$- \begin{bmatrix} H_1^{-1} \\ \vdots \\ H_n^{-1} \end{bmatrix} D^{-1} \begin{bmatrix} H_1^{-1} & \ldots & H_n^{-1} \end{bmatrix} \begin{bmatrix} \omega_1 H_1 \alpha_1^\circ \\ \vdots \\ \omega_n H_n \alpha_n^\circ \end{bmatrix},$$

or

$$\alpha_i - \alpha_i^\circ = H_i^{-1} \left(\sum_j \omega_j H_j^{-1} \right)^{-1} \left(\omega - \beta - \sum_j \omega_j \alpha_j^\circ \right).$$

The value function follows immediately from the definition of $\bar{a}(\omega)$ given by (23). □

A.5 Amplification and dampening

Proposition 8. *Let $\alpha^*(\mu, \Sigma)$ be the equilibrium production network under (μ, Σ) and let $W(\alpha, \mu, \Sigma)$ be the welfare of the household under the network α. The change in welfare after a change in beliefs from (μ, Σ) to (μ', Σ') satisfies the inequality*

$$\underbrace{\mathcal{W}(\mu', \Sigma') - \mathcal{W}(\mu, \Sigma)}_{\text{Change in welfare under a flexible network}} \geq \underbrace{W(\alpha^*(\mu, \Sigma), \mu', \Sigma') - W(\alpha^*(\mu, \Sigma), \mu, \Sigma)}_{\text{Change in welfare under a fixed network}}. \tag{38}$$

Proof. By definition, the change in welfare under the flexible network is

$$\mathcal{W}(\mu', \Sigma') - \mathcal{W}(\mu, \Sigma) = W(\alpha^*(\mu', \Sigma'), \mu', \Sigma') - W(\alpha^*(\mu, \Sigma), \mu, \Sigma),$$

and under the fixed network is

$$W(\alpha^*(\mu, \Sigma), \mu', \Sigma') - W(\alpha^*(\mu, \Sigma), \mu, \Sigma).$$

By Proposition 1, $\alpha^*(\mu', \Sigma')$ maximizes welfare under (μ', Σ') so that

$$W(\alpha^*(\mu', \Sigma'), \mu', \Sigma') \geq W(\alpha^*(\mu, \Sigma), \mu', \Sigma').$$

Combining the two expression gives the result. $\square$

B Additional proofs

This appendix contains proofs that are not included in the main appendix.

B.1 Proof of Corollary 1

Corollary 1. *For a fixed production network α, the following holds.*

1. *The impact of a change in expected TFP μ_i on the moments of log GDP is given by*

$$\frac{\partial \operatorname{E}[y]}{\partial \mu_i} = \omega_i, \quad \text{and} \quad \frac{\partial \operatorname{V}[y]}{\partial \mu_i} = 0.$$

2. *The impact of a change in volatility Σ_{ij} on the moments of log GDP is given by*

$$\frac{\partial \operatorname{E}[y]}{\partial \Sigma_{ij}} = 0, \quad \text{and} \quad \frac{\partial \operatorname{V}[y]}{\partial \Sigma_{ij}} = \omega_i \omega_j.$$

Proof. Equation (14) implies that $\frac{\partial \operatorname{E}[y(\alpha)]}{\partial \mu_i} = \beta^\top \mathcal{L}(\alpha) \mathbf{1}_i$. Since $P^\top C = WL = 1$ by the household's budget constraint, we need to show that $\beta^\top \mathcal{L}(\alpha) \mathbf{1}_i = P_i Q_i$ to complete the proof of the first result. From (A.3), we know that $P_i C_i = \beta_i$. Using Shepard's Lemma together with the marginal pricing equation (10), we can find the firm's factor demands equations

$$\begin{aligned} P_j X_{ij} &= \alpha_{ij} P_i Q_i, \\ L_i &= \left(1 - \sum_{j=1}^n \alpha_{ij}\right) P_i Q_i. \end{aligned} \tag{A.6}$$

Using these results, we can write the market clearing condition (11) as

$$P_i Q_i = \beta_i + \sum_{j=1}^n \alpha_{ji} P_j Q_j.$$

Solving the linear system implies

$$\beta^\top \mathcal{L}(\alpha) \mathbf{1}_i = P_i Q_i, \tag{A.7}$$

which proves the first part of the proposition. For the second part of the result, differentiating (14) with respect to Σ_{ij} and holding Σ symmetric yields

$$\frac{\partial \operatorname{V}[y(\alpha)]}{\partial \Sigma_{ij}} = \frac{1}{2} \beta^\top \mathcal{L}(\alpha) \left[\mathbf{1}_i \mathbf{1}_j^\top + \mathbf{1}_j \mathbf{1}_i^\top\right] [\mathcal{L}(\alpha)]^\top \beta,$$

which is the result. □

B.2 Proof of Corollary 3

Corollary 3. *If $\omega \in \operatorname{int}\mathcal{O}$, then the following holds.*

1. *An increase in the expected value μ_i or a decline in the variance Σ_{ii} leads to an increase in ω_j if i and j are global complements, and to a decline in ω_j if i and j are global substitutes.*

2. *An increase in the covariance Σ_{ij}, $i \neq j$, leads to a decline in ω_k if k is global complement with i and j, and to an increase in ω_k if k is global substitute with i and j.*

Proof. Combining (31) and (28), we can write $\frac{d\omega_j}{d\mu_i} = -\mathbf{1}_j^\top \mathcal{H}^{-1}\mathbf{1}_i = -\mathcal{H}^{-1}_{ji} = -\mathcal{H}^{-1}_{ij}$ since $\mathcal{H}$ is symmetric and inversion preserves symmetry. This proves point 1. For point 2, combining (31) and (29) yields $\frac{d\omega_j}{d\Sigma_{ii}} = (\rho-1)\,\omega_i \mathbf{1}_j^\top \mathcal{H}^{-1}\mathbf{1}_i = (\rho-1)\,\omega_i\mathcal{H}^{-1}_{ij}$, which is the result. For point 3, combining (31) and (29) yields

$$\frac{d\omega_k}{d\Sigma_{ij}} = \frac{1}{2}\,(\rho-1)\,\mathbf{1}_k^\top\mathcal{H}^{-1}\,(\omega_j\mathbf{1}_i + \omega_i\mathbf{1}_j) = \frac{1}{2}\,(\rho-1)\left(\omega_j\mathcal{H}^{-1}_{ik} + \omega_i\mathcal{H}^{-1}_{jk}\right),$$

which is the result. □

B.3 Proof of Lemma 5

Lemma 5. *An increase in the covariance Σ_{ij} induces stronger global substitution between i and j, in the sense that $\frac{\partial\mathcal{H}^{-1}_{ij}}{\partial\Sigma_{ij}} > 0$.*

Proof. Note that

$$\frac{\partial\mathcal{H}^{-1}_{ij}}{\partial\Sigma_{kl}} = \frac{1}{2}\,(\rho-1)\,\mathbf{1}_i^\top\mathcal{H}^{-1}\left(\mathbf{1}_k\mathbf{1}_l^\top + \mathbf{1}_l\mathbf{1}_k^\top\right)\mathcal{H}^{-1}\mathbf{1}_j,$$

where, if $k \neq l$, we differentiate with respect to Σ_{kl} and Σ_{lk} to preserve the symmetry of Σ and divide by two. In the special case with $i = k$ and $j = l$,

$$\begin{aligned}
\frac{\partial\mathcal{H}^{-1}_{kl}}{\partial\Sigma_{kl}} =& \frac{1}{2}\,(\rho-1)\,\mathbf{1}_k^\top\mathcal{H}^{-1}\left(\mathbf{1}_k\mathbf{1}_l^\top + \mathbf{1}_l\mathbf{1}_k^\top\right)\mathcal{H}^{-1}\mathbf{1}_l \\
=& \frac{1}{2}\,(\rho-1)\left\{\mathbf{1}_k^\top\mathcal{H}^{-1}\mathbf{1}_k\mathbf{1}_l^\top\mathcal{H}^{-1}\mathbf{1}_l + \mathbf{1}_k^\top\mathcal{H}^{-1}\mathbf{1}_l\mathbf{1}_k^\top\mathcal{H}^{-1}\mathbf{1}_l\right\} \\
=& \frac{1}{2}\,(\rho-1)\left\{\mathcal{H}^{-1}_{kk}\mathcal{H}^{-1}_{ll} + \left[\mathcal{H}^{-1}_{kl}\right]^2\right\} > 0.
\end{aligned}$$

The strict inequality holds because $\mathcal{H}^{-1}$ is a negative definite matrix. □

B.4 Proof of Lemma 6

Lemma 6. *Suppose that all input shares are (weak) local complements in the production of all goods, that is $\left[H_i^{-1}\right]_{kl} \leq 0$ for all i and all $k \neq l$. If $\alpha \in \operatorname{int} \mathcal{A}$, there exists a scalar $\bar{\Sigma} > 0$ such that if $\|\Sigma\| \leq \bar{\Sigma}$, all sectors are global complements, that is $\mathcal{H}_{ij}^{-1} < 0$ for all $i \neq j$.*

Proof. Consider the problem (23). In the internal equilibrium, $\alpha \in \operatorname{int} \mathcal{A}$, the first-order condition with respect to α_{ij} for this problem is

$$\omega_i \frac{\partial a_i}{\partial \alpha_{ij}} - \zeta_j \omega_i = 0 \Leftrightarrow \zeta_j = \frac{\partial a_i}{\partial \alpha_{ij}}, \tag{A.8}$$

where ζ is the vector of Lagrange multipliers associated with the constraint $\alpha^\top \omega - \omega + \beta = 0$.

Applying the envelope theorem to (23), we obtain

$$\nabla \bar{a} = a(\alpha) + (I - \alpha)\zeta.$$

Differentiation of this expression yields

$$\begin{aligned}\left(\nabla^2 \bar{a}\right)_{ij} = \frac{d^2 \bar{a}}{d\omega_i d\omega_j} &= \sum_{k=1}^n \frac{\partial a_i(\alpha_i)}{\partial \alpha_{ik}} \frac{d\alpha_{ik}}{d\omega_j} + (\mathbf{1}_i - \alpha_i)^\top \frac{d\zeta}{d\omega_j} - \zeta^\top \frac{d\alpha_i}{d\omega_j} \\ &= \sum_{k=1}^n \left(\frac{\partial a_i(\alpha_i)}{\partial \alpha_{ik}} - \zeta_k\right) \frac{d\alpha_{ik}}{d\omega_j} + (\mathbf{1}_i - \alpha_i)^\top \frac{d\zeta}{d\omega_j} \overset{(A.8)}{=} (\mathbf{1}_i - \alpha_i)^\top H_s \frac{d\alpha_s}{d\omega_j}.\end{aligned} \tag{A.9}$$

Note that from (A.8), $\frac{d\zeta}{d\omega_j} = H_s \frac{d\alpha_s}{d\omega_j}$ for any sector s. This implies, in particular, that $\frac{d\alpha_k}{d\omega_j} = H_k^{-1} H_s \frac{d\alpha_s}{d\omega_j}$ for any sector pair k, s.

Recall that $\omega^\top = \beta^\top \mathcal{L}(\alpha) \Leftrightarrow \alpha^\top \omega - \omega + \beta = 0$. Differentiating this expression with respect to ω_j, we get

$$\begin{aligned}&\sum_{k=1}^n \omega_k \frac{d\alpha_k}{d\omega_j} + \alpha^\top \mathbf{1}_j - \mathbf{1}_j = 0 \Leftrightarrow \sum_{k=1}^n \omega_k \left(H_k^{-1} H_s \frac{d\alpha_s}{d\omega_j}\right) + \alpha^\top \mathbf{1}_j - \mathbf{1}_j = 0 \Leftrightarrow \\ &H_s \frac{d\alpha_s}{d\omega_j} = \left(\sum_{k=1}^n \omega_k H_k^{-1}\right)^{-1} \left(I - \alpha^\top\right) \mathbf{1}_j.\end{aligned} \tag{A.10}$$

Combining this with (A.9), we get

$$\left(\nabla^2 \bar{a}\right)^{-1} = \left(\mathcal{L}^{-1} \left(\sum_{k=1}^n \omega_k H_k^{-1}\right)^{-1} \left(\mathcal{L}^\top\right)^{-1}\right)^{-1} = \mathcal{L}^\top \left(\sum_{k=1}^n \omega_k H_k^{-1}\right) \mathcal{L}, \tag{A.11}$$

where $\mathcal{L}^{-1} = I - \alpha$. Note that all elements of $\left(\nabla^2 \bar{a}\right)^{-1}$ are negative, $\left(\nabla^2 \bar{a}\right)^{-1} < 0$, if $\left[H_i^{-1}\right]_{kl} \leq 0$

for all i and all $k \neq l$. Indeed, in that case $H_i^{-1} \leq 0$ and H_i^{-1} has a strictly negative diagonal since H_i is negative definite. Furthermore, $\omega_i > 0$ for all i, and all elements of $\mathcal{L} = I + \alpha + \alpha^2 + \dots$ are positive since $\alpha \in \operatorname{int} \mathcal{A}$.

Next, consider $\mathcal{H}^{-1} = \left(\nabla^2 \bar{a} - (\rho - 1)\, \Sigma\right)^{-1}$. For $\Sigma = 0$ we have that $\mathcal{H}^{-1} = \left(\nabla^2 \bar{a}\right)^{-1} < 0$. Since the function $\left(\nabla^2 \bar{a} - (\rho - 1)\, \Sigma\right)^{-1}$ is continuous in Σ, it follows that there exists a threshold $\bar{\Sigma} > 0$ such that $\left(\nabla^2 \bar{a} - (\rho - 1)\, \Sigma\right)^{-1} < 0$ for $\|\Sigma\| \leq \bar{\Sigma}$, which is the result. □

B.5 Proof of Lemma 7

Lemma 7. *Suppose that all the TFP shifter functions $(a_1, \dots, a_n)$ take the form 2, with $\alpha_i^\circ = \alpha_j^\circ$ for all i, j, and that H_i^{-1} is of the form (33) for all i. If $\alpha \in \operatorname{int} \mathcal{A}$, there exists a scalar $\bar{\Sigma} > 0$ and a threshold $0 < \bar{s} < 1$ such that if $\|\Sigma\| \leq \bar{\Sigma}$ and $s > \bar{s}$, then all sectors are global substitutes, that is $\mathcal{H}_{ij}^{-1} > 0$ for all $i \neq j$.*

Proof. Imposing $\alpha_i^\circ = l^\circ$ and $H_i = H$ for all i, we can rewrite (24) as

$$\alpha_i(\omega) = l^\circ + \left(\sum_{j=1}^n \omega_j\right)^{-1} \left(\omega - \beta - l^\circ \sum_{j=1}^n \omega_j\right) = \left(\sum_{j=1}^n \omega_j\right)^{-1} (\omega - \beta).$$

Clearly, $\alpha_i(\omega) = l(\omega) > 0\ \forall i$. Using the Sherman-Morrison formula, we obtain

$$\mathcal{L} = (I - \alpha)^{-1} = \left(I - \mathbf{1} l^\top\right)^{-1} = I + \frac{\mathbf{1} l^\top}{1 - l^\top \mathbf{1}} = I + \frac{1}{1 - \sum_j l_j} \begin{bmatrix} l_1 & l_2 & \dots & l_n \\ l_1 & l_2 & & \vdots \\ \vdots & & \ddots & l_n \\ l_1 & \dots & l_{n-1} & l_n \end{bmatrix}. \tag{A.12}$$

Plugging H_i^{-1} from (33) into (A.11), we get

$$\left(\nabla^2 \bar{a}\right)_{im}^{-1} = \mathbf{1}_i^\top \mathcal{L}^\top \left(\sum_{k=1}^n \omega_k H_k^{-1}\right) \mathcal{L} \mathbf{1}_m = \left(\sum_{k=1}^n \omega_k\right) \sum_{k=1}^n \left(\mathcal{L}_{ki} \left(-\mathcal{L}_{km} + \frac{s}{n-1} \sum_{j \neq k} \mathcal{L}_{jm}\right)\right). \tag{A.13}$$

Notice that $\left(\nabla^2 \bar{a}\right)_{im}^{-1}$ is a continuous and strictly increasing function of s (the latter is true because for $\alpha \in \operatorname{int}\mathcal{A}$, all elements of $\mathcal{L}$ are positive). Furthermore, $\left(\nabla^2 \bar{a}\right)_{im}^{-1} < 0$ if $s = 0$. Using $\mathcal{L}$ given by (A.12), we get for $i \neq m$

$$
\begin{aligned}
\left(\nabla^2 \bar{a}\right)^{-1}_{im} &= \left(\sum_{k=1}^{n} \omega_k\right) \left[\mathcal{L}_{mi} \left(-\mathcal{L}_{mm} + \frac{s}{n-1} \sum_{j \neq m} \mathcal{L}_{jm}\right)\right. \\
&\quad \left. + \mathcal{L}_{ii} \left(-\mathcal{L}_{im} + \frac{s}{n-1} \sum_{j \neq i} \mathcal{L}_{jm}\right) + \sum_{k \neq m,i} \left(\mathcal{L}_{ki} \left(-\mathcal{L}_{km} + \frac{s}{n-1} \sum_{j \neq k} \mathcal{L}_{jm}\right)\right)\right] \\
&= \left(\sum_{k=1}^{n} \omega_k\right) \left[\frac{l_i}{1 - \sum_j l_j} \left(-1 - \frac{(1-s)\, l_m}{1 - \sum_j l_j}\right)\right. \\
&\quad \left. + \left(1 + \frac{l_i}{1 - \sum_j l_j}\right) \left(\frac{s}{n-1} - \frac{(1-s)\, l_m}{1 - \sum_j l_j}\right) + \frac{l_i\,(n-2)}{1 - \sum_j l_j} \left(\frac{s}{n-1} - \frac{(1-s)\, l_m}{1 - \sum_j l_j}\right)\right] \\
&\overset{s \to 1}{\to} \left(\sum_{k=1}^{n} \omega_k\right) \frac{1}{n-1} > 0. \qquad \text{(A.14)}
\end{aligned}
$$

It follows that we have global (strict) substitution for $\Sigma = 0$ and $s \to 1$, and global (strict) complementarity for $\Sigma = 0$ and $s = 0$. Furthermore, $\left(\nabla^2 \bar{a} + J_{\mathcal{E}}\right)^{-1}$ is continuous in s and Σ. Therefore, there exist thresholds $0 < \underline{s} \leq \bar{s} < 1$ and a threshold $\bar{\Sigma} > 0$ such that if $\|\Sigma\| \leq \bar{\Sigma}$ and $s > \bar{s}$ then $\left[\left(\nabla^2 \bar{a} + J_{\mathcal{E}}\right)^{-1}\right]_{im} > 0$ for $i \neq m$, and if $\|\Sigma\| \leq \bar{\Sigma}$ and $s < \underline{s}$ then $\left[\left(\nabla^2 \bar{a} + J_{\mathcal{E}}\right)^{-1}\right]_{im} < 0$ for $i \neq m$. □

B.6 Proof of Proposition 4

Proposition 4. *If $\alpha \in \operatorname{int} \mathcal{A}$, there exists a scalar $\bar{\Sigma} > 0$ such that if $\|\Sigma\| \leq \bar{\Sigma}$ the following holds.*

1. *(Complementarity) Suppose that input shares are local complements in the production of good i, that is $\left[H_i^{-1}\right]_{kl} < 0$ for all $k \neq l$. Then a beneficial change to k ($\partial \mathcal{E}_k / \partial \gamma > 0$) increases α_{ij} for all j.*

2. *(Substitution) Suppose that the conditions of Lemma 7 about the TFP shifters $(a_1, \ldots, a_n)$ hold. Then there exists a threshold $0 < \bar{s} < 1$ such that if $s > \bar{s}$, a beneficial change to k ($\partial \mathcal{E}_k / \partial \gamma > 0$) decreases α_{ij} for all i and all $j \neq k$, and increases α_{ik} for all i.*

Proof. From (A.10) we can write

$$
\left(\frac{d\alpha_{ij}}{d\omega}\right)^{\top} = \mathbf{1}_j^{\top} H_i^{-1} \left(\sum_{l=1}^{n} \omega_l H_l^{-1}\right)^{-1} \left(I - \alpha^{\top}\right).
$$

It follows that

$$\frac{d\alpha_{ij}}{d\gamma} = \left(\frac{d\alpha_{ij}}{d\omega}\right)^{\top}\frac{d\omega}{d\gamma} = \underbrace{\mathbf{1}_j^{\top} H_i^{-1}\left(\sum_{l=1}^{n}\omega_l H_l^{-1}\right)^{-1}\left(I-\alpha^{\top}\right)}_{\frac{d\alpha_{ij}}{d\omega^{\top}}}\underbrace{\left(-\mathcal{H}^{-1}\frac{\partial\mathcal{E}}{\partial\gamma}\right)}_{\frac{d\omega}{d\gamma}}. \tag{A.15}$$

If $\Sigma = 0$, then using (32) and (A.11) we find

$$\frac{d\alpha_{ij}}{d\gamma} = -\mathbf{1}_j^{\top} H_i^{-1}\left(I-\alpha\right)^{-1}\frac{\partial\mathcal{E}}{\partial\gamma}. \tag{A.16}$$

Recall that $(I-\alpha)^{-1} = I+\alpha+\alpha^2+\cdots > 0$ if $\alpha \in \operatorname{int}\mathcal{A}$. It follows that under local complementarity ($H_i^{-1} < 0$), a beneficial change to $\mathcal{E}_k$ increases α_{ij} for all i, j. Through the same argument as in the proof of Lemma 6, this holds for small $\|\Sigma\|$.

Suppose now that the TFP shifter functions $(a_1, \ldots, a_n)$ take the form 2 with $\alpha_i^{\circ} = l^{\circ}$, and $H_i = H$ is given by (33). Then $(I-\alpha)^{-1}$ is given by (A.12). Plugging those in (A.16), we obtain

$$\frac{d\alpha_{ij}}{d\gamma} = -\sum_{k\neq j}\left(\frac{s}{n-1} - \frac{(1-s)\,l_k}{1-\sum_m l_m}\right)\frac{\partial\mathcal{E}_k}{\partial\gamma} - \left(-1 - \frac{(1-s)\,l_j}{1-\sum_m l_m}\right)\frac{\partial\mathcal{E}_j}{\partial\gamma} \overset{s\to 1}{\to} -\frac{1}{n-1}\sum_{k\neq j}\frac{\partial\mathcal{E}_k}{\partial\gamma} + \frac{\partial\mathcal{E}_j}{\partial\gamma}.$$

The expression above is strictly negative if a beneficial shock hits sector $k \neq j$ ($\partial\mathcal{E}_k/\partial\gamma > 0$, $\partial\mathcal{E}_j/\partial\gamma = 0$), and is strictly positive if a beneficial shock hits sector j ($\partial\mathcal{E}_j/\partial\gamma > 0$, $\partial\mathcal{E}_k/\partial\gamma = 0$). Since the inequalities are strict, the same argument as in the proof of Lemma 7 applies and the results hold for $s > \bar{s} > 0$ and for small $\|\Sigma\|$. □

B.7 Proof of Corollary 5

Corollary 5. *Let γ denote either the mean μ_i or an element of the covariance matrix Σ_{ij}. The equilibrium response to a change in beliefs γ must satisfy*

$$\underbrace{\frac{d\,\mathrm{E}[y]}{d\gamma} - \frac{\partial\,\mathrm{E}[y]}{\partial\gamma}}_{\text{Excess response of } \mathrm{E}[y]} = \frac{1}{2}(\rho-1)\underbrace{\left(\frac{d\,\mathrm{V}[y]}{d\gamma} - \frac{\partial\,\mathrm{V}[y]}{\partial\gamma}\right)}_{\text{Excess response of } \mathrm{V}[y]}. \tag{41}$$

Proof. Suppose that the network is flexible. Then differentiating welfare with respect to γ implies

$$\frac{d\mathcal{W}}{d\gamma} = \frac{d\,\mathrm{E}[y]}{d\gamma} - \frac{1}{2}(\rho-1)\frac{d\,\mathrm{V}[y]}{d\gamma}.$$

Applying the envelope theorem to (21) implies

$$\frac{d\mathcal{W}}{d\gamma} = \frac{\partial\,\mathrm{E}[y]}{\partial\gamma} - \frac{1}{2}(\rho-1)\frac{\partial\,\mathrm{V}[y]}{\partial\gamma},$$

where partial derivatives, as usual, indicate that the network α is held fixed. Combining these two equations yields the result. □

B.8 Proof of Corollary 7

Corollary 7. *Suppose that $\omega \in \operatorname{int} \mathcal{O}$. There exists a threshold $\bar{\Sigma} < 0$ such that if $\Sigma_{kl} > \bar{\Sigma}$ for all k, l, then the following holds.*

1. *If all sectors are global complements with sector i, that is $\mathcal{H}_{ik}^{-1} < 0$ for $k \neq i$, then*

$$\frac{d\,\mathrm{E}[y]}{d\mu_i} > \omega_i, \quad \textit{and} \quad \frac{d\,\mathrm{V}[y]}{d\mu_i} > 0.$$

2. *If all sectors are global complements with sectors i and j, that is $\mathcal{H}_{ik}^{-1} < 0$ and $\mathcal{H}_{jk}^{-1} < 0$ for $k \neq i, j$, then*

$$\frac{d\,\mathrm{E}[y]}{d\Sigma_{ij}} < 0, \quad \textit{and} \quad \frac{d\,\mathrm{V}[y]}{d\Sigma_{ij}} < \omega_i\omega_j.$$

Proof. From part 1 of Proposition 7 and using (28),

$$\frac{d\,\mathrm{E}[y]}{d\mu_i} = \omega_i - (\rho - 1)\sum_{jk} \omega_j \Sigma_{jk} \mathcal{H}_{ik}^{-1}, \quad \text{and} \quad \frac{d\,\mathrm{V}[y]}{d\mu_i} = -2\sum_{jk} \omega_j \Sigma_{jk} \mathcal{H}_{ik}^{-1}.$$

Since $\mathcal{H}^{-1}$ is negative definite, $\mathcal{H}_{ii}^{-1} < 0$. Therefore, if $\mathcal{H}_{ik}^{-1} < 0$ for $k \neq i$, then there exists a threshold $\bar{\Sigma} < 0$ such that if $\Sigma_{jk} > \bar{\Sigma}$ for all j, k, then $\sum_{j,k} \omega_j \Sigma_{jk} \mathcal{H}_{ik}^{-1} < 0$ (recall that $\Sigma_{jj} > 0$ for all j). Therefore, $\frac{d\,\mathrm{E}[y]}{d\mu_i} > \omega_i$ and $\frac{d\,\mathrm{V}[y]}{d\mu_i} > 0$.

Using part 2 of Proposition 7 and (29), we can follow analogous steps to show that $\frac{d\,\mathrm{E}[y]}{d\Sigma_{ii}} < 0$ and $\frac{d\,\mathrm{V}[y]}{d\Sigma_{ii}} < \omega_i^2$ under the same conditions. Finally, from part 2 of Proposition 7 we have

$$\frac{d\,\mathrm{E}[y]}{d\Sigma_{ij}} = \frac{1}{2}(\rho - 1)^2 \sum_{lk} \left(\omega_l \omega_j \Sigma_{lk} \mathcal{H}_{ik}^{-1} + \omega_l \omega_i \Sigma_{lk} \mathcal{H}_{jk}^{-1}\right),$$

and

$$\frac{d\,\mathrm{V}[y]}{d\Sigma_{ij}} = \omega_i\omega_j + (\rho - 1)\sum_{lk} \left(\omega_l \omega_j \Sigma_{lk} \mathcal{H}_{ik}^{-1} + \omega_l \omega_i \Sigma_{lk} \mathcal{H}_{jk}^{-1}\right).$$

Since $\mathcal{H}^{-1}$ is negative definite, $\mathcal{H}_{ii}^{-1}, \mathcal{H}_{jj}^{-1} < 0$. Therefore, if $\mathcal{H}_{ik}^{-1} < 0$ and $\mathcal{H}_{jk}^{-1} < 0$ for $k \neq i$ and $k \neq j$, then then there exists a threshold $\bar{\Sigma} < 0$ such that if $\Sigma_{jk} > \bar{\Sigma}$ $\forall j$, then $\sum_{lk} \left(\omega_l \omega_j \Sigma_{lk} \mathcal{H}_{ik}^{-1} + \omega_l \omega_i \Sigma_{lk} \mathcal{H}_{jk}^{-1}\right) < 0$ (recall that $\Sigma_{jj} > 0$ for all j). Therefore, $\frac{d\,\mathrm{E}[y]}{d\Sigma_{ij}} < 0$ and $\frac{d\,\mathrm{V}[y]}{d\Sigma_{ij}} < \omega_i\omega_j$. □

B.9 Proof of Corollary 8

Corollary 8. *Suppose that $\omega \in \operatorname{int} \mathcal{O}$. Then there exist thresholds $\underline{\Sigma} > 0$ and $\bar{\Sigma} > 0$ such that,*

1. *If all sectors are global substitutes with sector i, that is $\mathcal{H}^{-1}_{ik} > 0$ for $k \neq i$, and sector i is not too risky while other sectors are sufficiently risky in the sense that $\Sigma_{ji} < \underline{\Sigma}$ for all j and $\Sigma_{jk} > \bar{\Sigma}$ for all $j, k \neq i$, then*

$$\frac{d\,\mathrm{E}[y]}{d\mu_i} < \omega_i, \quad \text{and} \quad \frac{d\,\mathrm{V}[y]}{d\mu_i} < 0.$$

2. *If all sectors are global substitutes with sectors i and j, that is $\mathcal{H}^{-1}_{ik} > 0$ and $\mathcal{H}^{-1}_{jk} > 0$ for $k \neq i, j$, and sectors i and j are not too risky while other sectors are sufficiently risky in the sense that $\Sigma_{li} < \underline{\Sigma}$ and $\Sigma_{lj} < \underline{\Sigma}$ for all l, and $\Sigma_{lk} > \bar{\Sigma}$ for all $l, k \neq i$ and $l, k \neq j$, then*

$$\frac{d\,\mathrm{E}[y]}{d\Sigma_{ij}} > 0, \quad \text{and} \quad \frac{d\,\mathrm{V}[y]}{d\Sigma_{ij}} > \omega_i\omega_j.$$

Proof. From part 1 of Proposition 7 and using (28),

$$\frac{d\,\mathrm{E}[y]}{d\mu_i} = \omega_i - (\rho - 1)\sum_{jk}\omega_j\Sigma_{jk}\mathcal{H}^{-1}_{ik}, \quad \text{and} \quad \frac{d\,\mathrm{V}[y]}{d\mu_i} = -2\sum_{jk}\omega_j\Sigma_{jk}\mathcal{H}^{-1}_{ik}.$$

Since $\mathcal{H}^{-1}$ is negative definite, $\mathcal{H}^{-1}_{ii} < 0$. Therefore, if $\mathcal{H}^{-1}_{ik} > 0$ for $k \neq i$, then there exist thresholds $\underline{\Sigma} > 0$ and $\bar{\Sigma} > 0$ such that if $\Sigma_{jk} > \bar{\Sigma}$ for all $j, k \neq i$, and $\Sigma_{ji} < \underline{\Sigma}$ for all j, then $\sum_{j,k}\omega_j\Sigma_{jk}\mathcal{H}^{-1}_{ik} > 0$. Therefore, $\frac{d\,\mathrm{E}[y]}{d\mu_i} < \omega_i$ and $\frac{d\,\mathrm{V}[y]}{d\mu_i} < 0$.

Using part 2 of Proposition 7 and (29), we can follow analogous steps to show that $\frac{d\,\mathrm{E}[y]}{d\Sigma_{ii}} > 0$ and $\frac{d\,\mathrm{V}[y]}{d\Sigma_{ii}} > \omega_i^2$ under the same conditions.

Finally, from part 3 of Proposition 7 we have

$$\frac{d\,\mathrm{E}[y]}{d\Sigma_{ij}} = \frac{1}{2}(\rho - 1)^2\sum_{lk}\left(\omega_l\omega_j\Sigma_{lk}\mathcal{H}^{-1}_{ik} + \omega_l\omega_i\Sigma_{lk}\mathcal{H}^{-1}_{jk}\right),$$

and

$$\frac{d\,\mathrm{V}[y]}{d\Sigma_{ij}} = \omega_i\omega_j + (\rho - 1)\sum_{lk}\left(\omega_l\omega_j\Sigma_{lk}\mathcal{H}^{-1}_{ik} + \omega_l\omega_i\Sigma_{lk}\mathcal{H}^{-1}_{jk}\right).$$

Since $\mathcal{H}^{-1}$ is negative definite, $\mathcal{H}^{-1}_{ii}, \mathcal{H}^{-1}_{jj} < 0$. Therefore, if $\mathcal{H}^{-1}_{ik} > 0$ and $\mathcal{H}^{-1}_{jk} > 0$ for $k \neq i, j$, then there exist thresholds $\underline{\Sigma} > 0$ and $\bar{\Sigma} > 0$ such that if $\Sigma_{lk} > \bar{\Sigma}$ for all $l, k \neq i$ and $l, k \neq j$, and $\Sigma_{li}, \Sigma_{lj} < \underline{\Sigma}$ for all l, then $\sum_{lk}\left(\omega_l\omega_j\Sigma_{lk}\mathcal{H}^{-1}_{ik} + \omega_l\omega_i\Sigma_{lk}\mathcal{H}^{-1}_{jk}\right) > 0$. Therefore, $\frac{d\,\mathrm{E}[y]}{d\Sigma_{ij}} > 0$ and $\frac{d\,\mathrm{V}[y]}{d\Sigma_{ij}} > \omega_i\omega_j$. □

C Microfoundation for the "one technique" restriction

In the main text, we made the ad hoc assumption that each sector can only adopt one production technique. Without this restriction, a large number of production techniques might be adopted and, after the shock ε is realized, only the technique that is best suited to this specific realization of ε would produce. In practice, we believe that several frictions might prevent this type of behavior. For instance, information frictions might make it impossible to redirect demand to the plant with the best technique after the shock is realized. Alternatively, engineers might be needed to explore how to set up a production technique, and there might be economies of scales pushing firms to adopt the same technique to save on engineering costs.

In this appendix, we propose one possible microfoundation for the "one technique" restriction. This microfoundation relies on decentralized trade for goods and on an information friction that prevents buyers from targeting specific producers. To describe this microfoundation, we first go over the economic agents in this environment. As in the main text, we still assume that there are n sectors/goods, but we are now explicit about the firms that operate within a sector. Specifically, in each sector $i \in \{1, \ldots, n\}$ there is a continuum of firms indexed by $l \in [0, 1]$. Each firm l operates a plant that can produce using a single production technique $\alpha_i^l \in \mathcal{A}_i$. We assume that physical restrictions, such as available factory space, prevent a plant for adopting multiple techniques. Different firms/plants in the same sector are however free to adopt different techniques.

Transactions between buyers (the household and intermediate firms) and sellers are conducted through shoppers. These shoppers are sent out by the buyers to meet sellers and negotiate terms of trade. Each shopper is atomistic, can purchase a measure one of goods and is matched with a seller at random. It follows that if in the market for good i there is a total demand of Q_i, each producer l is matched with a mass $Q_i dl$ of shoppers. Importantly, we assume that shoppers do not observe anything about the producers before they meet, and so cannot direct their search in any way.

If a shopper from firm m in sector j (or from the household) meets producer l in sector i, they agree to trade at a price $\tilde{P}_{il}^{jm}$ through a protocol described below.[1] From these prices we can compute the effective price paid by a firm in j (or by the household) for goods i. Since m sends a continuum of shoppers to all producers in sector i, the effective price it pays is equal to the average price

$$\tilde{P}_i^{jm} = \int_0^1 \tilde{P}_{il}^{jm} dl.$$

Individual prices $\tilde{P}_{il}^{jm}$ are set by splitting the joint surplus of the match through Nash bargaining. Specifically, if we denote the marginal benefit to the buyer of acquiring good i as B_i^{jm}, then the transaction price is such that the surplus of the seller is equal to a fraction $0 \leq \varsigma \leq 1$ of the total surplus. That is to say,

$$\tilde{P}_{il}^{jm} - K_i\left(\alpha_i^l, \left\{\tilde{P}_k^{il}\right\}_{k \in \{1,\ldots,n\}}\right) = \varsigma\left(B_i^{jm} - K_i\left(\alpha_i^l, \left\{\tilde{P}_k^{il}\right\}_{k \in \{1,\ldots,n\}}\right)\right), \tag{A.17}$$

[1]For notational convenience, let $j = 0$ denote the household and assume that there is a unit mass of "sub-households" indexed by $m \in [0, 1]$.

where $K_i\left(\alpha_i^l, \left\{\tilde{P}_k^{il}\right\}_{k\in\{1,\ldots,n\}}\right)$ is the unit cost of producer l in sector i under a chosen technique α_i^l.

From this last equation, we can write the technique choice problem of firm l. Since techniques are chosen before uncertainty is realized, we must average (A.17) across all states of the world, taking into account the stochastic discount factor of the household and the varying demand (mass of shoppers) for the good. It follows that firm l in sector i picks a production technique α_i^l to maximize

$$\mathrm{E}\left[\Lambda \sum_{j=0}^{n} Q_{ji} dl \int_0^1 \varsigma \left(B_i^{jm} - K_i\left(\alpha_i^l, \left\{\tilde{P}_k^{il}\right\}_{k\in\{1,\ldots,n\}}\right)\right) dm\right],$$

where $Q_{ji}dl$ denotes the demand from sector j for goods produced by firm l in sector i. Since α_i^l only affects this expression through K_i, this maximization problem is equivalent to minimizing

$$\mathrm{E}\left[\Lambda Q_i K_i\left(\alpha_i^l, \left\{\tilde{P}_k^{il}\right\}_{k\in\{1,\ldots,n\}}\right)\right], \tag{A.18}$$

where $Q_i = \sum_{j=0}^{n} Q_{ji}$ is total demand for sector i. Notice that this technique choice problem would be the same as the one described by (9) in the main text if the vector of input prices did not depend on the specific buying firm l and if all prices were equal to unit costs. To get that outcome, we now take the limit $\varsigma \to 0$, so that the bargaining power of the sellers goes to zero. In that case, (A.17) implies that

$$\tilde{P}_{il}^{jm} = K_i\left(\alpha_i^l, \left\{\tilde{P}_k^{il}\right\}_{k\in\{1,\ldots,n\}}\right),$$

and so $\tilde{P}_{il}^{jm}$ does not depend on the identity of the buyer, i.e. on j or m. It follows that effective demand $\tilde{P}_k^{il} = \int_0^1 \tilde{P}_{ks}^{il} ds$ does not depend on i and l, and we therefore can write $\tilde{P}_k^{il} = P_k$. Finally, this implies that the cost minimization problem (A.18) does not depend on the specific identity l of the firm. Given that the TFP shifter function is log concave, all firms in sector i therefore make the same technique choice α_i, have the same unit cost $K_i(\alpha_i, P)$ where $P = (P_1, \ldots, P_n)$, and that all prices are equal to unit cost, as in the model in the main text.

D Extension of Proposition 3 to binding constraints

We can straightforwardly extend Proposition 3 to handle the case in which some of the constraints $\omega_i \geq 0$ bind with strictly positive Lagrange multipliers.[2] Note that these Domar weights ω_i will not respond to a marginal change in beliefs. We still assume, however, that the constraint $1 \geq \omega^\top(\mathbf{1} - \bar{\alpha})$ is slack.

Formally, define a set of indices $\mathcal{I} = \{i : \omega_i \geq \beta_i \text{ binds}\}$. For $j \notin \mathcal{I}$, $\omega_j > \beta_j$ and for $j \in \mathcal{I}$,

[2] Thus, we rule out cases in which $\omega_i = \beta_i$ but the Lagrange multiplier corresponding to the $\omega_i \geq \beta_i$ is also zero. At such points, the derivative of ω_i with respect to a change in beliefs might be not defined.

$\omega_j = \beta_j$. Define an $\hat{n} \times 1$ vector ω^{nb} that contains elements ω_i such that $i \notin \mathcal{I}$, and an $(n - \hat{n}) \times 1$ vector ω^b that contains elements ω_i such that $i \in \mathcal{I}$, where $0 \leq \hat{n} \leq n$. Then ω^{nb} is implicitly given by the first-order conditions of (26), i.e.

$$\mathcal{E}^{nb} + \frac{d\bar{a}\left(\omega\right)}{d\omega^{nb}} = 0,$$

where $\mathcal{E}^{nb}$ is an $\hat{n}\times 1$ vector obtained by deleting elements $k \in \mathcal{I}$ from $\mathcal{E}$, and $\frac{d\bar{a}(\omega)}{d\omega^{nb}}$ is an $\hat{n} \times 1$ vector obtained by differentiating (23) with respect to ω^{nb}. Applying the implicit function theorem, we can write

$$\frac{d\omega^{nb}}{d\gamma} = -\left(\mathcal{H}^{nb}\right)^{-1} \times \frac{\partial \mathcal{E}^{nb}}{\partial \gamma}. \tag{A.19}$$

In the expression above,

$$\mathcal{H}^{nb} = \frac{d\bar{a}\left(\omega\right)}{d\omega^{nb} d\left(\omega^{nb}\right)^{\top}} - \left(\rho - 1\right)\Sigma^{nb},$$

where $\frac{d\bar{a}(\omega)}{d\omega^{nb}d\left(\omega^{nb}\right)^{\top}}$ is an $\hat{n} \times \hat{n}$ Hessian matrix obtained by differentiating (23) twice with respect to ω^{nb}, and $\frac{\partial \mathcal{E}^{nb}}{\partial \omega^{nb}} = -\left(\rho - 1\right)\Sigma^{nb}$, where Σ^{nb} is an $\hat{n} \times \hat{n}$ matrix obtained by deleting columns and rows $k \in \mathcal{I}$ from Σ. Finally, compute $\frac{\partial \mathcal{E}^{nb}}{\partial \gamma}$ for $\gamma = \mu_i$ and Σ_{ij}:

$$\begin{aligned}
\frac{\partial \mathcal{E}^{nb}}{\partial \mu_i} &= \mathbf{1}_i^{nb}, \\
\frac{\partial \mathcal{E}^{nb}}{\partial \Sigma_{ij}} &= -\frac{1}{2}\left(\rho - 1\right)\left(\omega_j \mathbf{1}_i^{nb} + \omega_i \mathbf{1}_j^{nb}\right),
\end{aligned}$$

where $\mathbf{1}_i^{nb}$ is an $\hat{n} \times 1$ vector obtained by deleting elements $k \in \mathcal{I}$ from $\mathbf{1}_i$. In particular, if $i \in \mathcal{I}$ then all elements of $\mathbf{1}_i^{nb}$ are zeros. Consequently, if $i \in \mathcal{I}$ then a change in μ_i or Σ_{ii} has no impact on ω^{nb}.

E More details about the regressions of Section 9

In this section, we provide more details about the regressions presented in Tables I and II. The firm-level production network data comes from the Factset Revere database and covers the period from 2003 to 2016. We limit the sample to relationships that have lasted at least five years. The IV estimates remain significant when relationships of other lengths are considered. The firm-level uncertainty data comes from Alfaro et al. (2019) and was downloaded from Nicholas Bloom's website at `https://nbloom.people.stanford.edu`. We thank the authors for sharing their data. Alfaro et al. (2019) describes how the data is constructed in detail, and we only include here a summary of how the instruments are computed. The instruments are created by first computing the industry-level sensitivity to each aggregate shock c, where c is either the price of oil, one of seven exchange rates, the yield on 10-year U.S. Treasury Notes and the economic policy uncertainty index of Baker et al. (2016). As Alfaro et al. (2019) explain, "for firm i in industry j, sensitivity$_j^c = \beta_j^c$ is estimated as follows

$$r_{i,t}^{riskadj} = \alpha_j + \sum_c \beta_j^c \cdot r_t^c + \epsilon_{i,t},$$

where $r_{i,t}^{riskadj}$ is the daily risk-adjusted return of firm i, r_t^c is the change in the price of commodity c, and α_j is industry j's intercept. [...] Estimating the main coefficients of interest, β_j^c, at the SIC 3-digit level (instead of at the firm-level) reduces the role of idiosyncratic noise in firm-level returns, increasing the precision of the estimates. [...] We allow these industry-level sensitivities to be time-varying by estimating them using 10-year rolling windows of daily data." The instruments $z_{i,t-1}^c$ are then computed as follows:

$$z_{i,t-1}^c = \left|\beta_{j,t-1}^c\right| \cdot \Delta\sigma_{t-1}^c,$$

where $\Delta\sigma_{t-1}^c$ denotes the volatility of the aggregate variable c. As a result, instruments vary on the 3-digit SIC industry-by-year level. As in Alfaro et al. (2019), we also include in the IV regressions the first moments associated with each aggregate series c ("1st moment $10\text{IV}_{i,t-1}$" in Tables I and II) to isolate the impact of changes in their second moment alone. Note that we control for year×customer×supplier industry (2-digit SIC) fixed effects in Tables I and II. Therefore, instruments and control variables used in columns (2) and (3) exhibit nontrivial variation within fixed-effects bins. At the same time, such rich fixed effects allow us to compare how a given customer firm in a given year reacts to different volatility shocks hitting its suppliers within the same 2-digit SIC industry.

F Alternative specifications for the distribution of ε

The parametrization of the shock process ε that we use in the model is common in the uncertainty literature (see for instance, Bloom et al., 2018), but has the implication that a change in the covariance matrix Σ has a direct impact on expected GDP $\mathrm{E}[Y]$, and so can affect decisions even when the household is risk neutral ($\rho = 0$). This happens because the mean of a log-normal variable like GDP is an increasing function of the variance of the underlying normal distribution. A common approach used by many papers is to undo this effect by removing half of the variance from the mean of the normal distribution. Such a change is, however, problematic in our setup.

In this appendix, we first describe that in our setting there is no parametrization of ε such that 1) ε is normally distributed, 2) Σ does not affect decisions when $\rho = 0$, and 3) the distribution of ε does not depend on endogenous objects. We then consider a version of the model in which the distribution of ε is such that changes in Σ do not affect any decision when $\rho = 0$. This specification is however conceptually problematic as the distribution of ε depends on endogenous equilibrium objects. Finally, we consider a specification in which we adjust the mean of ε so that changes in Σ have no effect on $\mathrm{E}\left[e^{\varepsilon}\right]$. In that case, the expectation of firm-level TFP shocks is unaffected by Σ; however, the expectation of macroeconomic aggregates, e.g. $\mathrm{E}\left[Y\right]$, still depend on Σ.

F.1 How to parametrize ε so that a risk-neutral household does not respond to uncertainty

In this subsection we describe how ε must be parametrized so that a risk-neutral household ($\rho = 0$) does not change its behavior in response to changes in uncertainty Σ. For that purpose, it is useful to go back to central equations of the model that hold whenever ε is normally distributed. Hulten's theorem implies that for any given network α, log GDP is given by $y = \omega\left(\alpha\right)^{\top}\left(\varepsilon + a\left(\alpha\right)\right)$, where $\omega\left(\alpha\right)$ is the vector of Domar weights. Together with CRRA preferences, this implies that the social planner's problem can be written as

$$\begin{aligned}\mathcal{W} &\equiv \max_{\alpha\in\mathcal{A}} \mathrm{E}\left[y\left(\alpha\right)\right] - \frac{1}{2}\left(\rho - 1\right)\mathrm{V}\left[y\left(\alpha\right)\right] \\ &= \max_{\alpha\in\mathcal{A}} \omega\left(\alpha\right)^{\top}\left(\mathrm{E}\left[\varepsilon\right] + a\left(\alpha\right)\right) - \frac{1}{2}\left(\rho - 1\right)\omega\left(\alpha\right)^{\top}\mathrm{V}\left[\varepsilon\right]\omega\left(\alpha\right).\end{aligned}$$

In the benchmark model we have $\varepsilon \sim \mathcal{N}\left(\mu, \Sigma\right)$ and clearly Σ matters for the planner's decisions when $\rho = 0$. Suppose instead that $\varepsilon \sim \mathcal{N}\left(\mu - \frac{1}{2}B, \Sigma\right)$ where B is some quantity that can depend on Σ and that would make α^* invariant to Σ when $\rho = 0$. Plugging in the planner's problem, we find

$$\begin{aligned}\mathcal{W} &= \max_{\alpha\in\mathcal{A}} \omega\left(\alpha\right)^{\top}\left(\mu - \frac{1}{2}B + a\left(\alpha\right)\right) - \frac{1}{2}\left(\rho - 1\right)\omega\left(\alpha\right)^{\top}\Sigma\omega\left(\alpha\right) \\ &= \max_{\alpha\in\mathcal{A}} \omega\left(\alpha\right)^{\top}\left(\mu + a\left(\alpha\right)\right) - \frac{1}{2}\rho\omega\left(\alpha\right)^{\top}\Sigma\omega\left(\alpha\right) + \frac{1}{2}\omega\left(\alpha\right)^{\top}\left(\Sigma\omega\left(\alpha\right) - B\right).\end{aligned}$$

For Σ to have no influence when $\rho = 0$ we therefore need the last term to be zero, which requires $B = \Sigma\omega\left(\alpha\right)$. In other words, this requires that the distribution of firm-level TFP shocks itself depends on endogenous equilibrium objects, namely the Domar weights $\omega\left(\alpha\right)$. This is problematic for at least two important reasons. First, we cannot think of a good reason why the distribution of productivity shocks that affect one industry would depend on the production technique chosen by another industry. Why that dependence would operate through Domar weights is also unclear. Second, the parametrization $\varepsilon \sim \mathcal{N}\left(\mu - \frac{1}{2}\Sigma\omega\left(\alpha\right), \Sigma\right)$ potentially introduces an externality in the economy: when deciding on its input shares α_i, firm i is modifying the TFP process of all other firms in the economy. This would create a gap between the efficient and the equilibrium allocations.

F.2 A model in which risk considerations are absent when $\rho = 0$

Here, we propose a distribution for ε such that 1) changes in Σ do not affect decisions when the household is risk neutral ($\rho = 0$), and 2) the equilibrium coincides with the solution to the planner's problem. Note that simply setting $B = \Sigma\omega(\alpha)$ does not accomplish this because of the externalities mentioned above.

Specifically, we assume that

$$\varepsilon \sim \mathcal{N}\left(\mu - g\left(\alpha, \alpha^*, \Sigma\right), \Sigma\right), \tag{A.20}$$

where

$$g\left(\alpha, \alpha^*, \Sigma\right) = \frac{1}{2}\Sigma\mathcal{L}\left(\alpha^*\right)^\top \beta + \frac{1}{2}\left(\alpha - \alpha^*\right)^\top \mathcal{L}\left(\alpha^*\right)\Sigma\mathcal{L}\left(\alpha^*\right)^\top \beta. \tag{A.21}$$

The term α^* in this expression is the equilibrium network, so that in equilibrium we have $g\left(\alpha^*, \alpha^*, \Sigma\right) = \frac{1}{2}\Sigma\mathcal{L}\left(\alpha^*\right)^T \beta$. When making decisions, the representative firm in sector i chooses $\alpha_i = \left(\alpha_{i1}, \ldots, \alpha_{in}\right)$ but takes α^* as given.

A few comments are in order. First, this specification implies that the distribution of shocks depends on endogenous equilibrium objects. This is clearly conceptually problematic, but it is, as we have discussed above, required for the result. We are not arguing that this specification is desirable or plausible. Our goal here is to explore the conditions under which decisions are unaffected by Σ under risk neutrality. Second, instead of assuming that g shifts the mean of ε, we could equivalently include it in the TFP shifter A. In that case, A would depend on equilibrium objects, unlike in the baseline specification. Third, the specification (A.20)–(A.21) differs from the one discussed above, $\varepsilon \sim \mathcal{N}\left(\mu - \frac{1}{2}\Sigma\omega\left(\alpha\right), \Sigma\right)$, which made the planner's problem unaffected by Σ under $\rho = 0$. Notice that both specification coincide *in equilibrium* but extra terms are required in (A.20)–(A.21) to ensure that the decentralized equilibrium allocation is efficient.

Once production techniques have been chosen and a specific realization of ε has been drawn, the distribution of ε has no impact on the economy. Therefore, Lemmas 1 and 2 also hold under this alternative specification (with $\mathrm{E}\left[p\left(\alpha^*\right)\right] = -\mathcal{L}\left(\alpha^*\right)\left(\mu - g\left(\alpha^*, \alpha^*, \Sigma\right) + a\left(\alpha^*\right)\right)$ in Lemma 2). Furthermore, as derived in Section F.1, the planner's objective is given by

$$\mathcal{W} \equiv \max_{\alpha \in \mathcal{A}} \underbrace{\beta^\top \mathcal{L}\left(\alpha\right)\left(\mu + a\left(\alpha\right)\right)}_{\log \mathrm{E}[Y(\alpha)]} - \frac{1}{2}\rho \underbrace{\beta^\top \mathcal{L}\left(\alpha\right)\Sigma\mathcal{L}\left(\alpha\right)^\top \beta}_{\mathrm{V}[y(\alpha)]}.$$

Following the same steps as in the main text, we can establish that there exists a unique solution to the planner's problem. One can also establish that there exists a unique equilibrium and it is efficient. The proof is analogous to that of Proposition 1. The key step in that proof is to show that the first-order conditions of the planner's problem and of the firm's problem coincide. This is indeed the case. For the planner's problem, we have

$$\frac{\partial a_i}{\partial \alpha_i} + \mathcal{L}(\alpha)(\mu + a(\alpha)) - \rho \mathcal{L}(\alpha) \Sigma \mathcal{L}(\alpha)^\top \beta + \chi_i^p - \gamma_i^p = 0. \tag{A.22}$$

Consider now the firm's problem. Combining (46) with (12), (44) and (45), we find that the problem of the representative firm in sector i can be written as

$$\begin{aligned}\alpha_i^* = \arg\min_{\alpha_i \in \mathcal{A}_i} \frac{1}{2}(\alpha_i - \alpha_i^*)^\top \mathcal{L}(\alpha^*) \Sigma \mathcal{L}(\alpha^*)^\top \beta - a(\alpha_i) - \alpha_i^\top \mathcal{L}(\alpha^*) \left(\mu - \frac{1}{2}\Sigma \mathcal{L}(\alpha^*)^\top \beta + a(\alpha^*)\right) \\ + \frac{1}{2}\left((\alpha_i - \mathbf{1}_i - (1-\rho)\beta)^\top \mathcal{L}(\alpha^*) + \mathbf{1}_i^\top\right) \Sigma \left((\alpha_i - \mathbf{1}_i - (1-\rho)\beta)^\top \mathcal{L}(\alpha^*) + \mathbf{1}_i^\top\right)^\top.\end{aligned}$$

Differentiating with respect to α_{ij} we can write the first-order conditions as

$$\begin{aligned}0 = \mathbf{1}_j^\top \mathcal{L}(\alpha^*) \Sigma \mathcal{L}(\alpha^*)^\top \beta - \frac{\partial a(\alpha_i)}{\partial \alpha_{ij}} - \mathbf{1}_j^\top \mathcal{L}(\alpha^*)(\mu + a(\alpha^*)) \\ + \left(\mathbf{1}_j^\top \mathcal{L}(\alpha^*)\right) \Sigma \left((\alpha_i - \mathbf{1}_i - (1-\rho)\beta)^\top \mathcal{L}(\alpha^*) + \mathbf{1}_i^\top\right)^\top - \chi_{ij}^e + \gamma_i^e,\end{aligned}$$

In equilibrium $\alpha = \alpha^*$ and so the above expression simplifies to

$$\frac{\partial a(\alpha_i^*)}{\partial \alpha_{ij}} + \mathbf{1}_j^\top \mathcal{L}(\alpha^*)(\mu + a(\alpha^*)) - \rho \beta^\top \mathcal{L}(\alpha^*) \Sigma \mathcal{L}(\alpha^*)^\top \mathbf{1}_j + \chi_{ij}^e - \gamma_i^e = 0,$$

which is equivalent to (A.22). Finally, the results in Sections 6 and 7 remain unchanged, with the only exception that $(\rho - 1)$ should replaced by ρ.

F.3 Making the expectation of firm-level TFP shocks independent of Σ

One specification for ε which is used in the literature is $\varepsilon \sim \mathcal{N}\left(\mu - \frac{1}{2}\text{diag}(\Sigma), \Sigma\right)$. This adjustment implies that the expected value of firm-level TFP $\text{E}[\exp(\varepsilon_i)] = \exp\left(\mu_i - \frac{1}{2}\Sigma_{ii} + \frac{1}{2}\Sigma_{ii}\right) = \exp(\mu_i)$ does not depend on Σ. Changes to Σ are therefore closer to pure changes in uncertainty. But, as follows from the discussion above, Σ still matters for decisions even when the household is risk neutral. Almost all our analytical results are unaffected by this change in specification. The only differences appear when we take derivatives with respect to Σ. In that case, the results need to be adapted to capture the impact of Σ on the expected value of ε.

Changes to quantitative results

We also investigate the implications of this change in specification for our quantitative model. To do so, we consider an alternative economy, denoted with tildes, in which

$$\tilde{\varepsilon} \sim \mathcal{N}\left(\tilde{\mu} - \frac{1}{2}\text{diag}\left(\tilde{\Sigma}\right), \tilde{\Sigma}\right).$$

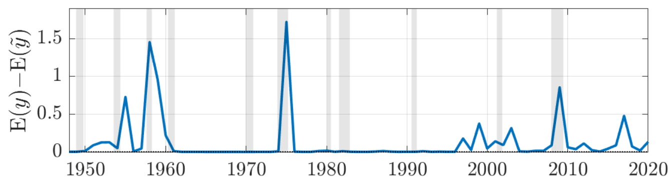
If we were to calibrate this economy, we would find that

$$\begin{aligned}\tilde{\mu}_t - \frac{1}{2}\text{diag}\left(\tilde{\Sigma}_t\right) &= \mu_t, \\ \tilde{\Sigma}_t &= \Sigma_t,\end{aligned}$$

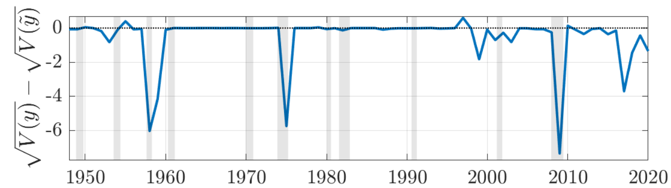
where μ_t and Σ_t are the mean and covariance of ε_t in our baseline calibration. That is because the calibration matches the vector of sectoral TFP perfectly. If we remove $\frac{1}{2}\Sigma_{ii}$ from the expectation of ε_i, the estimation would increase the expectation $\tilde{\mu}_t$ to compensate and match the data.

We reproduce the exercise in the left column of Figure 4 in the main text in this setup. This amounts to comparing the economy described above with an alternative in which the production network is chosen *as if* $\tilde{\Sigma}_t = 0$. The results are presented in Figure 1. Overall, we find that uncertainty has a larger impact on the economy in this setting than in the baseline model of Section 8. As in Figure 4 in the main text, the variance of log GDP is smaller in the baseline model. Expected log GDP is however quite different, with $\text{E}[y]$ larger in the baseline than in the alternative model. This is because the network in the alternative economy is not well adapted to the TFP process. In the alternative model, firms choose production techniques as if $\text{E}[\tilde{\varepsilon}] = \tilde{\mu}$, when in reality $\text{E}[\tilde{\varepsilon}] = \tilde{\mu} - \frac{1}{2}\text{diag}\tilde{\Sigma}$. This implies that firms ignore the fact that risky suppliers (i.e. those with high Σ_{ii}) are also less productive on average, which results in a decline in expected log GDP relative to the baseline model (first panel of Figure 1). Given that the alternative model performs worse than the baseline both in terms of $\text{E}[y]$ and $\text{V}[y]$, the welfare losses in the alternative model are substantial (third panel).

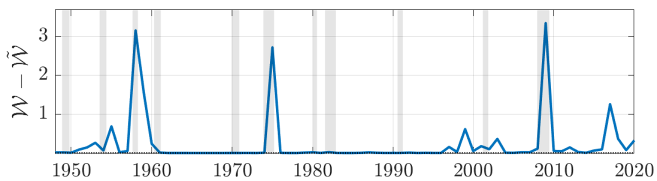
Figure 1. The role of uncertainty when $\varepsilon \sim \mathcal{N}\left(\mu - \frac{1}{2}\text{diag}\left(\Sigma\right), \Sigma\right)$



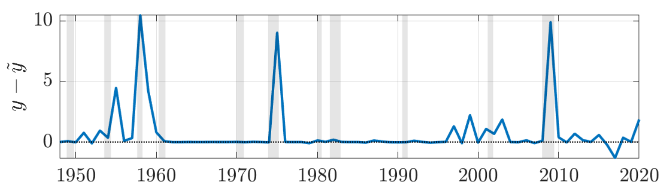
(a) Difference in expected log GDP [%]



(b) Difference in expected standard deviation of log GDP [%]



(c) Difference in expected welfare [%]



(d) Difference in realized log GDP [%]

Notes: The differences between the series implied by the baseline model with $\varepsilon \sim \mathcal{N}\left(\mu - \frac{1}{2}\text{diag}\left(\Sigma\right), \Sigma\right)$ (without tildes) and the "as if $\tilde{\Sigma}_t = 0$" alternative (with tildes). Both economies are hit by the same shocks that are filtered out from the TFP data under our baseline model. All differences are expressed in percentage terms.

G Approximated economy

Section G.1 considers the economy in which the cost of deviating from the ideal shares is large. All the proofs are in Section G.2.

G.1 Large costs of deviating from the ideal input shares

In this section, we consider an economy in which the cost of deviating from the ideal input shares α° is large. Let $a_i(\alpha_i) = \bar{\kappa} \times \hat{a}_i(\alpha_i)$, where $\hat{a}_i$ does not depend on $\bar{\kappa}$ and is such that $\hat{a}_i(\alpha_i^\circ) = 0$. Suppose that $\alpha_i^\circ \in \text{int}\,\mathcal{A}_i$. The parameter $\bar{\kappa} > 0$ captures how costly it is for the firms to deviate from α° in terms of TFP losses.[3]

Our goal is to characterize the economy when $\bar{\kappa} > 0$ is large. To do so, we use perturbation theory to express the equilibrium production network as a second-order approximation (Judd and Guu, 2001; Schmitt-Grohé and Uribe, 2004). More explicitly, let $\alpha(\bar{\kappa})$ denote the production network under a given cost shifter $\bar{\kappa}$, and consider the expansion

$$\alpha(\bar{\kappa}) = \alpha_{(0)} + \bar{\kappa}^{-1}\alpha_{(1)} + \bar{\kappa}^{-2}\alpha_{(2)} + O\left(\bar{\kappa}^{-3}\right), \tag{A.23}$$

where $\alpha_{(m)}$ denotes the mth order term. Notably, for a sufficiently large $\bar{\kappa}$, $\alpha(\bar{\kappa}) \in \operatorname{int}\mathcal{A}$ because $\alpha^{\circ} \in \operatorname{int}\mathcal{A}$ by assumption. We will provide expressions for these terms in what follows, but first it is convenient to rewrite some equilibrium quantities using the expansion (A.23).

Throughout, we will work with variables that are evaluated at the ideal input shares. We use the superscript $\circ$ to denote these quantities. For instance, $\mathcal{L}^{\circ} = (I - \alpha^{\circ})^{-1}$ is the Leontief inverse evaluated at α°, $\hat{H}_i^{\circ}$ is the Hessian of $\hat{a}_i$ at α_i°, and so on.

The following lemma provides approximate expressions for the Leontief inverse and the TFP shifter.

Lemma 11. *The following holds:*

1. *The Leontief inverse $\mathcal{L}$ can be written as $\mathcal{L} = \mathcal{L}^{\circ} + \bar{\kappa}^{-1}\mathcal{L}_{(1)} + \bar{\kappa}^{-2}\mathcal{L}_{(2)} + O\left(\kappa^{-3}\right)$, where*

$$\mathcal{L}_{(1)} = \mathcal{L}^{\circ}\alpha_{(1)}\mathcal{L}^{\circ}, \tag{A.24}$$

$$\mathcal{L}_{(2)} = \mathcal{L}^{\circ}\alpha_{(2)}\mathcal{L}^{\circ} + \mathcal{L}_{(1)}\alpha_{(1)}\mathcal{L}^{\circ}. \tag{A.25}$$

2. *The TFP shifter $\hat{a}_i$ can be written as $\hat{a}_i = \bar{\kappa}^{-2}\hat{a}_{i,(2)} + O\left(\bar{\kappa}^{-4}\right)$, where*

$$\hat{a}_{i,(2)} = \frac{1}{2}\alpha_{i,(1)}^{\top}\hat{H}_i^{\circ}\alpha_{i,(1)}, \tag{A.26}$$

   and where $\alpha_{i,(1)}^{\top}$ is the ith row of $\alpha_{(1)}$.

The first part of the lemma states that to a first order the Leontief inverse can be expressed as a deviation from $\mathcal{L}^{\circ}$ that is linear in $\alpha_{(1)}$. Naturally, the second-order term is linear in $\alpha_{(2)}$ and quadratic in $\alpha_{(1)}$. Its second part shows that as the production network moves away from α° the TFP loss associated with that move depends on the curvature of the TFP shifter function captured by $\hat{H}_i^{\circ}$. The zero-order term in the expression of $\hat{a}_i$ is zero because $\hat{a}_i(\alpha_i^{\circ}) = 0$, and the first-order term $\nabla\hat{a}_i(\alpha_i^{\circ}) = 0$ since the ideal shares maximize $\hat{a}_i$.[4]

[3] Throughout this section we assume that all the third derivatives of $\hat{a}_i$ are zero. Relaxing this assumption is straightforward but requires heavier notation that obscures the exposition of the mechanisms,

[4] The third-order term is zero by assumption that all the third derivatives of $\hat{a}_i$ are zero.

Approximated risk-adjusted prices

With these quantities in hand, we can derive an expression for the risk-adjusted price vector.

Lemma 12. *The risk-adjusted price vector $\mathcal{R}$ can be written as $\mathcal{R} = \mathcal{R}_{(0)} + \bar{\kappa}^{-1}\mathcal{R}_{(1)} + O\left(\kappa^{-2}\right)$, where*

$$\mathcal{R}_{(0)} = \mathcal{R}^{\circ} = \underbrace{-\mathcal{L}^{\circ}\mu}_{\mathrm{E}[p^{\circ}]} + \underbrace{(\rho-1)\,\mathcal{L}^{\circ}\Sigma\omega^{\circ}}_{\mathrm{Cov}(p^{\circ},\lambda^{\circ})} \tag{A.27}$$

is the risk-adjusted price vector (18) evaluated at the ideal input shares α°, and

$$\mathcal{R}_{(1)} = \underbrace{-\mathcal{L}_{(1)}\mu - \mathcal{L}^{\circ}\hat{a}_{(2)}}_{\textit{Change in } \mathrm{E}[p]} + \underbrace{(\rho-1)\,\mathcal{L}^{\circ}\Sigma\mathcal{L}_{(1)}^{\top}\beta + (\rho-1)\,\mathcal{L}_{(1)}\Sigma\left[\mathcal{L}^{\circ}\right]^{\top}\beta}_{\textit{Change in } \mathrm{Cov}(p,\lambda)}, \tag{A.28}$$

where $\mathcal{L}_{(1)}$ and $\hat{a}_{(2)}$ are given by (A.24) and (A.26).

Equation (A.27) shows that to a first-order approximation $\mathcal{R}$ is simply equal to its value at the ideal input shares α°. The second-order term (A.28) describes that deviating from α° impacts $\mathcal{R}$ in two ways. First, it changes the importance of different sectors as suppliers, as captured by $\mathcal{L}_{(1)}$, and it modifies the TFP losses associated with the choice of technique, captured by $\hat{a}_{(2)}$. Together, these two terms reflect the impact of α on the expected price vector $\mathrm{E}[p]$. Second, the change in α modifies the covariance between the stochastic discount factor and the price vector. The last two terms in (A.28) capture that channel.

Approximated production network and Domar weights

The following proposition provides a second-order approximation for the production network α.

Proposition 9. *If $\alpha \in \operatorname{int}\mathcal{A}$, the equilibrium input shares in sector i are approximately given by*

$$\alpha_i = \alpha_i^{\circ} + \bar{\kappa}^{-1}\underbrace{\left(\hat{H}_i^{\circ}\right)^{-1}\mathcal{R}_{(0)}}_{\alpha_{i,(1)}} + \bar{\kappa}^{-2}\underbrace{\left(\hat{H}_i^{\circ}\right)^{-1}\mathcal{R}_{(1)}}_{\alpha_{i,(2)}} + O\left(\bar{\kappa}^{-3}\right), \tag{A.29}$$

where $\mathcal{R}_{(0)}$ and $\mathcal{R}_{(1)}$ are given by (A.27) and (A.28).

This expression, together with the two preceding lemmas, provides a closed-form characterization of the equilibrium production network in terms of the primitives of the model as a second-order approximation. To understand its structure, recall from Lemma (2) that the equilibrium can be characterized as the fixed point of the self-map given by the right-hand side of (17). We can iterate on this self-map to find the equilibrium network. This involves starting from an initial risk-adjusted price vector, computing the optimal production network under these prices, computing the new risk-adjusted prices that correspond to that network, and restarting that process again until convergence.

Proposition 9 mimics this structure. From Equation (19), the inverse Hessian H_i^{-1} captures how risk-adjusted prices affect the input shares chosen by firm i. The term $\alpha_i^\circ + \bar{\kappa}^{-1}\left(\hat{H}_i^\circ\right)^{-1}\mathcal{R}^\circ$ in (A.29) therefore corresponds to the firms optimal decision under $\mathcal{R}^\circ$, which acts as the initial vector for the iterations. The following term in (A.29) captures the next step in the iteration process. The quantity $\mathcal{R}_{(1)}$ corresponds to the response of $\mathcal{R}$ to the first round decision of the firms. Multiplying this quantity by $\left(\hat{H}_i^\circ\right)^{-1}$ then provides the reaction of the firms to this change in $\mathcal{R}$.

One implication of Proposition 9 is that further iterations of the equilibrium mapping have a decreasing impact on the production network. We see from the second term in (A.29) that the reaction of the firms to the original risk-adjusted prices is of order $\bar{\kappa}^{-1}$, while the second iteration term is of order $\bar{\kappa}^{-2}$. This suggests that when $\bar{\kappa}$ is large the first few rounds of iteration should provide an accurate picture of the production network.

Combining the expressions for the Leontief inverse (A.24) and (A.25) with the expressions for $\alpha_{(1)}$ and $\alpha_{(2)}$ given by (A.29), it is straightforward to derive a formula for the Domar weights.

Corollary 9. *The equilibrium vector of Domar weights is*

$$\omega = \omega^\circ + \bar{\kappa}^{-1}\underbrace{\mathcal{L}_{(1)}^\top\beta}_{\omega_{(1)}} + \kappa^{-2}\underbrace{\mathcal{L}_{(2)}^\top\beta}_{\omega_{(2)}} + O\left(\bar{\kappa}^{-3}\right), \tag{A.30}$$

where $\omega^\circ = (\mathcal{L}^\circ)^\top\beta$, and $\mathcal{L}_{(1)}$ and $\mathcal{L}_{(2)}$ are given by (A.24) and (A.25).

Using (A.29) and (A.30), one can also explicitly show that the approximate formula (35) shown at the end of Section 6.1 is accurate if $\bar{\kappa}$ is sufficiently large.

Corollary 10. *The approximation of the Domar weight vector (35) is accurate if $\bar{\kappa}$ is large, such that*

$$-\left[\mathcal{H}^\circ\right]^{-1}\mathcal{E}^\circ = \bar{\kappa}^{-1}\omega_{(1)} + O\left(\bar{\kappa}^{-2}\right). \tag{A.31}$$

Approximated GDP

Recall that moments of log GDP are given by (14). Using our results above, it is straightforward to derive approximate expressions for $\mathrm{E}\,y$ and $\mathrm{V}\,y$.

Corollary 11. *In equilibrium, the mean and variance of log GDP are*

$$\mathrm{E}\,y = (\omega^\circ)^\top\mu + \bar{\kappa}^{-1}\left(\omega_{(1)}^\top\mu + (\omega^\circ)^\top\hat{a}_{(2)}\right) + \bar{\kappa}^{-2}\left(\omega_{(2)}^\top\mu + \omega_{(1)}^\top\hat{a}_{(2)}\right) + O\left(\bar{\kappa}^{-3}\right)$$

and

$$\mathrm{V}\,y = (\omega^\circ)\,\Sigma\omega^\circ + 2\bar{\kappa}^{-1}(\omega^\circ)^\top\Sigma\omega_{(1)} + \bar{\kappa}^{-2}\left(2\,(\omega^\circ)^\top\Sigma\omega_{(2)} + \omega_{(1)}^\top\Sigma\omega_{(1)}\right) + O\left(\bar{\kappa}^{-3}\right).$$

Response of the production network to changes in beliefs

We now provide two results that describe, as closed-form second-order approximation, the response of the production network to changes in beliefs.

The following corollary provides a closed-form expression for $\frac{d\alpha_i}{d\mu_k}$ as a second-order approximation.

Corollary 12. *The impact of an increase in μ_k on the network is given by*

$$\frac{d\alpha_i}{d\mu_k} = \bar{\kappa}^{-1}\left(\hat{H}_i^{\circ}\right)^{-1} \underbrace{\left(\frac{\partial \mathcal{R}_{(0)}}{\partial \mu_k} + \bar{\kappa}^{-1}\frac{\partial \mathcal{R}_{(1)}}{\partial \mu_k}\right)}_{\text{Response of } \mathcal{R} \text{ with fixed network}} + \bar{\kappa}^{-2}\left(\hat{H}_i^{\circ}\right)^{-1} \underbrace{\left(\sum_{lm} \frac{d\alpha_{lm,(1)}}{d\mu_k} \times \frac{\partial \mathcal{R}_{(1)}}{\partial \alpha_{lm,(1)}}\right)}_{\text{Impact of the network on } \mathcal{R}} + O\left(\bar{\kappa}^{-3}\right), \tag{A.32}$$

where the impact of μ_k on $\mathcal{R}$ taking the network fixed is given by

$$\frac{\partial \mathcal{R}_{(0)}}{\partial \mu_k} + \bar{\kappa}^{-1}\frac{\partial \mathcal{R}_{(1)}}{\partial \mu_k} = -\mathcal{L}^{\circ}\mathbf{1}_k - \bar{\kappa}^{-1}\mathcal{L}_{(1)}\mathbf{1}_k, \tag{A.33}$$

and the change in $\mathcal{R}$ through the response of the network is given by

$$\frac{d\alpha_{lm,(1)}}{d\mu_k} \times \frac{\partial \mathcal{R}_{(1)}}{\partial \alpha_{lm,(1)}} = -\mathbf{1}_m^{\top}\left(\hat{H}_l^{\circ}\right)^{-1}\mathcal{L}^{\circ}\mathbf{1}_k \times (\rho-1)\,\mathcal{L}^{\circ}\Sigma\left(\mathcal{L}^{\circ}\mathbf{1}_l\mathbf{1}_m^{\top}\mathcal{L}^{\circ}\right)^{\top}\beta. \tag{A.34}$$

The following corollary provides a similar result for the derivative of the network with respect to Σ_{op} for $o \neq p$. Its results also apply for the case $o = p$ if all the terms $\frac{1}{2}\left(\mathbf{1}_o\mathbf{1}_p^{\top} + \mathbf{1}_p\mathbf{1}_o^{\top}\right)$ are replaced by $\mathbf{1}_o\mathbf{1}_o^{\top}$, as in this case there is no need to take two derivatives to preserve the symmetry of Σ.

Corollary 13. *The impact of an increase in Σ_{op} on the network is given by*

$$\frac{d\alpha_i}{d\Sigma_{op}} = \bar{\kappa}^{-1}\left(\hat{H}_i^{\circ}\right)^{-1} \underbrace{\left(\frac{\partial \mathcal{R}_{(0)}}{\partial \Sigma_{op}} + \bar{\kappa}^{-1}\frac{\partial \mathcal{R}_{(1)}}{\partial \Sigma_{op}}\right)}_{\text{Response of } \mathcal{R} \text{ with fixed network}} + \bar{\kappa}^{-2}\left(\hat{H}_i^{\circ}\right)^{-1} \underbrace{\left(\sum_{lm} \frac{d\alpha_{lm,(1)}}{d\Sigma_{op}} \times \frac{\partial \mathcal{R}_{(1)}}{\partial \alpha_{op,(1)}}\right)}_{\text{Impact of the network on } \mathcal{R}} + O\left(\bar{\kappa}^{-3}\right), \tag{A.35}$$

where the impact of Σ_{op} on $\mathcal{R}$ taking the network fixed is given by

$$\begin{aligned}\frac{\partial \mathcal{R}_{(0)}}{\partial \Sigma_{op}} + \bar{\kappa}^{-1}\frac{\partial \mathcal{R}_{(1)}}{\partial \Sigma_{op}} &= (\rho-1)\,\mathcal{L}^{\circ}\left[\frac{1}{2}\left(\mathbf{1}_o\mathbf{1}_p^{\top} + \mathbf{1}_p\mathbf{1}_o^{\top}\right)\right]\omega^{\circ} \\ &+ \bar{\kappa}^{-1}\left[(\rho-1)\,\mathcal{L}^{\circ}\left[\frac{1}{2}\left(\mathbf{1}_o\mathbf{1}_p^{\top} + \mathbf{1}_p\mathbf{1}_o^{\top}\right)\right]\mathcal{L}_{(1)}^{\top}\beta + (\rho-1)\,\mathcal{L}_{(1)}\left[\frac{1}{2}\left(\mathbf{1}_o\mathbf{1}_p^{\top} + \mathbf{1}_p\mathbf{1}_o^{\top}\right)\right]\left(\mathcal{L}^{\circ}\right)^{\top}\beta\right],\end{aligned}$$

and the change in $\mathcal{R}$ through the response of the network is given by

$$\frac{d\alpha_{lm,(1)}}{d\Sigma_{op}} \times \frac{\partial \mathcal{R}_{(1)}}{\partial \alpha_{lm,(1)}} = (\rho - 1)\left(\hat{H}_i^{\circ}\right)^{-1} \mathcal{L}^{\circ} \left[\frac{1}{2}\left(\mathbf{1}_o \mathbf{1}_p^{\top} + \mathbf{1}_p \mathbf{1}_o^{\top}\right)\right] \omega^{\circ} \times (\rho - 1)\, \mathcal{L}^{\circ} \Sigma \left(\mathcal{L}^{\circ} \mathbf{1}_l \mathbf{1}_m^{\top} \mathcal{L}^{\circ}\right)^{\top} \beta. \tag{A.36}$$

G.2 Proofs related to the approximation

Proof of Lemma 11

Lemma 11. *The following holds:*

1. *The Leontief inverse $\mathcal{L}$ can be written as $\mathcal{L} = \mathcal{L}^{\circ} + \bar{\kappa}^{-1}\mathcal{L}_{(1)} + \bar{\kappa}^{-2}\mathcal{L}_{(2)} + O\left(\kappa^{-3}\right)$, where*

$$\mathcal{L}_{(1)} = \mathcal{L}^{\circ}\alpha_{(1)}\mathcal{L}^{\circ}, \tag{A.24}$$

$$\mathcal{L}_{(2)} = \mathcal{L}^{\circ}\alpha_{(2)}\mathcal{L}^{\circ} + \mathcal{L}_{(1)}\alpha_{(1)}\mathcal{L}^{\circ}. \tag{A.25}$$

2. *The TFP shifter $\hat{a}_i$ can be written as $\hat{a}_i = \bar{\kappa}^{-2}\hat{a}_{i,(2)} + O\left(\bar{\kappa}^{-4}\right)$, where*

$$\hat{a}_{i,(2)} = \frac{1}{2}\alpha_{i,(1)}^{\top}\hat{H}_i^{\circ}\alpha_{i,(1)}, \tag{A.26}$$

and where $\alpha_{i,(1)}^{\top}$ is the ith row of $\alpha_{(1)}$.

Proof. The rules of differentiation for a matrix inverse imply that

$$\frac{d\mathcal{L}(\alpha)}{d\alpha_{kl}} = \frac{d\left((I-\alpha)^{-1}\right)}{d\alpha_{kl}} = (I-\alpha)^{-1}\frac{d\alpha}{d\alpha_{kl}}(I-\alpha)^{-1} = \mathcal{L}(\alpha)\,\mathbf{1}_k\mathbf{1}_l^{\top}\mathcal{L}(\alpha),$$

and

$$\left[\frac{d\mathcal{L}(\alpha)}{d\alpha_{kl}}\right]_{ij} = \mathbf{1}_i^{\top}\mathcal{L}(\alpha)\,\mathbf{1}_k\mathbf{1}_l^{\top}\mathcal{L}(\alpha)\,\mathbf{1}_j = \left[\mathcal{L}(\alpha)\right]_{ik}\left[\mathcal{L}(\alpha)\right]_{lj}.$$

By Taylor's theorem, and using the notation (A.23) for the deviation from α°, we find

$$\begin{aligned}
\mathcal{L}_{ij} = \mathcal{L}_{ij}^{\circ} &+ \sum_{kl}\mathcal{L}_{ik}^{\circ}\left(\bar{\kappa}^{-1}\alpha_{kl,(1)}\right)\mathcal{L}_{lj}^{\circ} + \sum_{kl}\mathcal{L}_{ik}^{\circ}\left(\bar{\kappa}^{-2}\alpha_{kl,(2)}\right)\mathcal{L}_{lj}^{\circ} \\
&+ \frac{1}{2}\left\{\sum_{kl}\left(\sum_{sm}\mathcal{L}_{is}^{\circ}\left(\bar{\kappa}^{-1}\alpha_{sm,(1)}\right)\mathcal{L}_{mk}^{\circ}\right)\left(\bar{\kappa}^{-1}\alpha_{kl,(1)}\right)\mathcal{L}_{lj}^{\circ}\right. \\
&+ \left.\sum_{kl}\mathcal{L}_{ik}^{\circ}\left(\bar{\kappa}^{-1}\alpha_{kl,(1)}\right)\left(\sum_{sm}\mathcal{L}_{ls}^{\circ}\left(\bar{\kappa}^{-1}\alpha_{sm,(1)}\right)\mathcal{L}_{mj}^{\circ}\right)\right\} + O\left(\bar{\kappa}^{-3}\right),
\end{aligned}$$

which can be written in the matrix form as

$$\mathcal{L} = \mathcal{L}^{\circ} + \bar{\kappa}^{-1}\mathcal{L}^{\circ}\alpha_{(1)}\mathcal{L}^{\circ} + \bar{\kappa}^{-2}\left(\mathcal{L}^{\circ}\alpha_{(2)}\mathcal{L}^{\circ} + \mathcal{L}^{\circ}\alpha_{(1)}\mathcal{L}^{\circ}\alpha_{(1)}\mathcal{L}^{\circ}\right) + O\left(\bar{\kappa}^{-3}\right).$$

This completes the proof of the first part of the lemma. For the second part, we can write

$$\hat{a}_i(\alpha_i) \approx \hat{a}_i(\alpha_i^\circ) + (\alpha_i - \alpha_i^\circ)^\top \begin{pmatrix} \frac{\partial \hat{a}_i}{\partial \alpha_{i1}} \\ \vdots \\ \frac{\partial \hat{a}_i}{\partial \alpha_{in}} \end{pmatrix}_{\alpha_i = \alpha_i^\circ} + \frac{1}{2}(\alpha_i - \alpha_i^\circ)^\top \hat{H}_i^\circ (\alpha_i - \alpha_i^\circ) + O\left((\alpha_i - \alpha_i^\circ)^4\right).$$

The first term is equal to zero by the definition of $\hat{a}_i$. The second term is equal to zero since α_i° maximizes $\hat{a}_i$. The third-order term is zero by assumption that all the third derivatives of $\hat{a}_i$ are zero. Using the notation (A.23) for the deviation from α°, we can write

$$\hat{a}_i(\alpha_i) \approx \frac{1}{2}\bar{\kappa}^{-2}\alpha_{i,(1)}^\top \hat{H}_i^\circ \alpha_{i,(1)} + O\left(\bar{\kappa}^{-4}\right),$$

which is the result. □

Proof of Lemma 12

Lemma 12. *The risk-adjusted price vector* $\mathcal{R}$ *can be written as* $\mathcal{R} = \mathcal{R}_{(0)} + \bar{\kappa}^{-1}\mathcal{R}_{(1)} + O\left(\kappa^{-2}\right)$, *where*

$$\mathcal{R}_{(0)} = \mathcal{R}^\circ = \underbrace{-\mathcal{L}^\circ \mu}_{\mathrm{E}[p^\circ]} + \underbrace{(\rho - 1)\mathcal{L}^\circ \Sigma \omega^\circ}_{\mathrm{Cov}(p^\circ, \lambda^\circ)} \tag{A.27}$$

is the risk-adjusted price vector (18) *evaluated at the ideal input shares* α°, *and*

$$\mathcal{R}_{(1)} = \underbrace{-\mathcal{L}_{(1)}\mu - \mathcal{L}^\circ \hat{a}_{(2)}}_{\textit{Change in } \mathrm{E}[p]} + \underbrace{(\rho - 1)\mathcal{L}^\circ \Sigma \mathcal{L}_{(1)}^\top \beta + (\rho - 1)\mathcal{L}_{(1)}\Sigma [\mathcal{L}^\circ]^\top \beta}_{\textit{Change in } \mathrm{Cov}(p, \lambda)}, \tag{A.28}$$

where $\mathcal{L}_{(1)}$ *and* $\hat{a}_{(2)}$ *are given by* (A.24) *and* (A.26).

Proof. We can write $\mathcal{R}$ as

$$\mathcal{R} = -\mathcal{L}(\alpha)(\mu + \bar{\kappa}\hat{a}(\alpha)) + (\rho - 1)\mathcal{L}(\alpha)\Sigma[\mathcal{L}(\alpha)]^\top \beta, \tag{A.37}$$

or

$$\begin{aligned} \mathcal{R} = &-\left(\mathcal{L}^\circ + \bar{\kappa}^{-1}\mathcal{L}_{(1)} + O\left(\bar{\kappa}^{-2}\right)\right)\left(\mu + \bar{\kappa} \times \bar{\kappa}^{-2}\hat{a}_{(2)} + \bar{\kappa}O\left(\bar{\kappa}^{-3}\right)\right) \\ &+ (\rho - 1)\left(\mathcal{L}^\circ + \bar{\kappa}^{-1}\mathcal{L}_{(1)} + O\left(\bar{\kappa}^{-2}\right)\right)\Sigma\left(\mathcal{L}^\circ + \bar{\kappa}^{-1}\mathcal{L}_{(1)} + O\left(\bar{\kappa}^{-2}\right)\right)^\top \beta, \end{aligned}$$

where we used the expressions in Lemma 11. Grouping terms yields the result. □

Proof of Proposition 9

Proposition 9. *If $\alpha \in \operatorname{int} \mathcal{A}$, the equilibrium vector of input shares in sector i is*

$$\alpha_i = \alpha_i^\circ + \bar{\kappa}^{-1} \underbrace{\left(\hat{H}_i^\circ\right)^{-1} \mathcal{R}_{(0)}}_{\alpha_{i,(1)}} + \bar{\kappa}^{-2} \underbrace{\left(\hat{H}_i^\circ\right)^{-1} \mathcal{R}_{(1)}}_{\alpha_{i,(2)}} + O\left(\bar{\kappa}^{-3}\right), \tag{A.29}$$

where $\mathcal{R}_{(0)}$ and $\mathcal{R}_{(1)}$ are given by (A.27) and (A.28).

Proof. Since $\alpha^\circ \in \operatorname{int} \mathcal{A}$, when $\bar{\kappa}$ is large enough the equilibrium network α is also in the interior of $\mathcal{A}$. From (19), that equilibrium is then fully described by the first-order conditions of the problem (17), such that

$$\frac{\partial \hat{a}_i\left(\alpha_i\right)}{\partial \alpha_{ij}} = \bar{\kappa}^{-1} \times \mathcal{R}_j\left(\alpha\right), \tag{A.38}$$

and where $\mathcal{R}_j\left(\alpha\right)$ is given by (A.37). We can write the left-hand side of this equation as

$$\begin{aligned} \frac{\partial \hat{a}_i\left(\alpha_i\right)}{\partial \alpha_{ij}} = & \left(\frac{\partial \hat{a}_i}{\partial \alpha_{ij}}\right)_{\alpha_i = \alpha_i^\circ} + \left(\alpha_i - \alpha_i^\circ\right)^\top \begin{pmatrix} \frac{\partial^2 \hat{a}_i}{\partial \alpha_{i1} \partial \alpha_{ij}} \\ \cdots \\ \frac{\partial^2 \hat{a}_i}{\partial \alpha_{in} \partial \alpha_{ij}} \end{pmatrix}_{\alpha_i = \alpha_i^\circ} \\ & + \frac{1}{2}\left(\alpha_i - \alpha_i^\circ\right)^\top \begin{pmatrix} \frac{\partial^3 \hat{a}_i}{\partial \alpha_{i1}^2 \partial \alpha_{ij}} & \cdots & \frac{\partial^3 \hat{a}_i}{\partial \alpha_{i1} \partial \alpha_{in} \partial \alpha_{ij}} \\ \cdots & \cdots & \cdots \\ \frac{\partial^3 \hat{a}_i}{\partial \alpha_{i1} \partial \alpha_{in} \partial \alpha_{ij}} & \cdots & \frac{\partial^3 \hat{a}_i}{\partial \alpha_{in}^2 \partial \alpha_{ij}} \end{pmatrix}_{\alpha_i = \alpha_i^\circ} \left(\alpha_i - \alpha_i^\circ\right) + O\left(\left(\alpha_i - \alpha_i^\circ\right)^3\right). \end{aligned}$$

The first term in the right-hand side of this expression is zero since α° maximizes $\hat{a}_i$. The third term is also zero given our assumption that the third derivatives of $\hat{a}_i$ are zero. Using the notation from (A.23), we can therefore write

$$\frac{\partial \hat{a}_i\left(\alpha_i\right)}{\partial \alpha_{ij}} = \left(\bar{\kappa}^{-1} \alpha_{i,(1)} + \bar{\kappa}^{-2} \alpha_{i,(2)}\right)^\top \hat{H}_i^\circ \mathbf{1}_j + O\left(\bar{\kappa}^{-3}\right).$$

Combing with the right-hand side of (A.38) and the expressions of Lemma 12, we find

$$\left(\bar{\kappa}^{-1} \alpha_{i,(1)} + \bar{\kappa}^{-2} \alpha_{i,(2)}\right)^\top \hat{H}_i^\circ \mathbf{1}_j + O\left(\bar{\kappa}^{-3}\right) = \bar{\kappa}^{-1}\left(\mathcal{R}_{j,(0)} + \bar{\kappa}^{-1} \mathcal{R}_{j,(1)} + O\left(\kappa^{-2}\right)\right).$$

Since this expression must be valid for all $\bar{\kappa}$, we find that

$$\alpha_{i,1} = \left(\hat{H}_i^\circ\right)^{-1} \mathcal{R}^\circ$$

and

$$\alpha_{i,2} = \left(\hat{H}_i^\circ\right)^{-1} \mathcal{R}_{(1)},$$

which is the result. ☐

Proof of Corollary 9

Corollary 9. *The equilibrium vector of Domar weights is*

$$\omega = \omega^\circ + \bar{\kappa}^{-1} \underbrace{\mathcal{L}_{(1)}^\top \beta}_{\omega_{(1)}} + \kappa^{-2} \underbrace{\mathcal{L}_{(2)}^\top \beta}_{\omega_{(2)}} + O\left(\bar{\kappa}^{-3}\right), \tag{A.30}$$

where $\omega^\circ = \left(\mathcal{L}^\circ\right)^\top \beta$, and $\mathcal{L}_{(1)}$ and $\mathcal{L}_{(2)}$ are given by (A.24) and (A.25).

Proof. This corollary immediately follows from the definition of the Domar weights, $\omega = \mathcal{L}\left(\alpha\right)^\top \beta$. ☐

Proof of Corollary 10

Corollary 10. *The approximation of the Domar weight vector (35) is accurate if $\bar{\kappa}$ is large, such that*

$$-\left[\mathcal{H}^\circ\right]^{-1} \mathcal{E}^\circ = \bar{\kappa}^{-1} \omega_{(1)} + O\left(\bar{\kappa}^{-2}\right). \tag{A.31}$$

Proof. Using (18) and (32), we can express the left-hand side of (A.31) as

$$-\left[\mathcal{H}^\circ\right]^{-1} \mathcal{E}^\circ = -\left[\nabla^2 \bar{a}\left(\omega^\circ\right) - \left(\rho - 1\right)\Sigma\right]^{-1}\left(\mu - \left(\rho - 1\right)\Sigma\omega^\circ\right).$$

Using (A.11) to express $\nabla^2 \bar{a}\left(\omega^\circ\right)$, we get

$$\begin{aligned}
-\left[\mathcal{H}^\circ\right]^{-1} \mathcal{E}^\circ &= -\left[\left(\mathcal{L}^\circ\right)^{-1}\left(\sum_{k=1}^{n} \omega_k^\circ \left(H_k^\circ\right)^{-1}\right)^{-1}\left[\left(\mathcal{L}^\circ\right)^\top\right]^{-1} - \left(\rho - 1\right)\Sigma\right]^{-1}\left(\mu - \left(\rho - 1\right)\Sigma\omega^\circ\right) \\
&= -\left[\bar{\kappa}\left(\mathcal{L}^\circ\right)^{-1}\left(\sum_{k=1}^{n} \omega_k^\circ \left(\hat{H}_k^\circ\right)^{-1}\right)^{-1}\left[\left(\mathcal{L}^\circ\right)^\top\right]^{-1} - \left(\rho - 1\right)\Sigma\right]^{-1}\left(\mu - \left(\rho - 1\right)\Sigma\omega^\circ\right) \\
&= -\bar{\kappa}^{-1}\left(\mathcal{L}^\circ\right)^\top \sum_{k=1}^{n}\left(\omega_k^\circ \left(\hat{H}_k^\circ\right)^{-1}\right)\mathcal{L}^\circ\left(\mu - \left(\rho - 1\right)\Sigma\omega^\circ\right) + O\left(\bar{\kappa}^{-2}\right).
\end{aligned}$$

From (A.29), the expression above can be rewritten as

$$-\left[\mathcal{H}^\circ\right]^{-1} \mathcal{E}^\circ = \bar{\kappa}^{-1}\left(\mathcal{L}^\circ\right)^\top \sum_{k=1}^{n} \omega_k^\circ \alpha_{k,(1)} + O\left(\bar{\kappa}^{-2}\right) = \bar{\kappa}^{-1}\mathcal{L}_{(1)}^\top \beta + O\left(\bar{\kappa}^{-2}\right),$$

where $\mathcal{L}_{(1)}$ is given by (A.24) and $\omega_{(1)} = \mathcal{L}_{(1)}^{\top}\beta$ by (A.30). □

Proof of Corollary 11

Corollary 11. *In equilibrium, the mean and variance of log GDP are*

$$\mathrm{E}\, y = (\omega^{\circ})^{\top} \mu + \bar{\kappa}^{-1} \left(\omega_{(1)}^{\top}\mu + (\omega^{\circ})^{\top} \hat{a}_{(2)}\right) + \bar{\kappa}^{-2} \left(\omega_{(2)}^{\top}\mu + \omega_{(1)}^{\top}\hat{a}_{(2)}\right) + O\left(\bar{\kappa}^{-3}\right)$$

and

$$\mathrm{V}\, y = (\omega^{\circ})\, \Sigma\omega^{\circ} + 2\bar{\kappa}^{-1} (\omega^{\circ})^{\top} \Sigma\omega_{(1)} + \bar{\kappa}^{-2} \left(2 (\omega^{\circ})^{\top} \Sigma\omega_{(2)} + \omega_{(1)}^{\top}\Sigma\omega_{(1)}\right) + O\left(\bar{\kappa}^{-3}\right).$$

Proof. These expressions are straightforward to derive by plugging in (14) expressions (A.26) and (A.30). □

Proof of Corollary 12

Corollary 12. *The impact of an increase in μ_k on the network is given by*

$$\frac{d\alpha_i}{d\mu_k} = \bar{\kappa}^{-1} \left(\hat{H}_i^{\circ}\right)^{-1} \underbrace{\left(\frac{\partial \mathcal{R}_{(0)}}{\partial \mu_k} + \bar{\kappa}^{-1}\frac{\partial \mathcal{R}_{(1)}}{\partial \mu_k}\right)}_{\text{Response of } \mathcal{R} \text{ with fixed network}} + \bar{\kappa}^{-2} \left(\hat{H}_i^{\circ}\right)^{-1} \underbrace{\left(\sum_{lm} \frac{\partial \mathcal{R}_{(1)}}{\partial \alpha_{lm,(1)}} \frac{d\alpha_{lm,(1)}}{d\mu_k}\right)}_{\text{Impact of the network on } \mathcal{R}} + O\left(\bar{\kappa}^{-3}\right), \tag{A.32}$$

where the impact of μ_k on $\mathcal{R}$ taking the network fixed is given by

$$\frac{\partial \mathcal{R}_{(0)}}{\partial \mu_k} + \bar{\kappa}^{-1}\frac{\partial \mathcal{R}_{(1)}}{\partial \mu_k} = -\mathcal{L}^{\circ}\mathbf{1}_k - \bar{\kappa}^{-1}\mathcal{L}_{(1)}\mathbf{1}_k, \tag{A.33}$$

and the change in $\mathcal{R}$ through the response of the network is given by

$$\frac{d\alpha_{lm,(1)}}{d\mu_k} \times \frac{\partial \mathcal{R}_{(1)}}{\partial \alpha_{lm,(1)}} = -\mathbf{1}_m^{\top} \left(\hat{H}_l^{\circ}\right)^{-1} \mathcal{L}^{\circ}\mathbf{1}_k \times (\rho - 1)\, \mathcal{L}^{\circ}\Sigma \left(\mathcal{L}^{\circ}\mathbf{1}_l\mathbf{1}_m^{\top}\mathcal{L}^{\circ}\right)^{\top} \beta. \tag{A.34}$$

Proof. Differentiating (A.23) with respect to μ_k yields

$$\frac{d\alpha_i}{d\mu_k} = \bar{\kappa}^{-1}\frac{d\alpha_{i,(1)}}{d\mu_k} + \bar{\kappa}^{-2}\frac{d\alpha_{i,(2)}}{d\mu_k} + O\left(\bar{\kappa}^{-3}\right).$$

Using the expressions for the first and second-order terms given by the Proposition 9, we find

$$\frac{d\alpha_i}{d\mu_k} = \bar{\kappa}^{-1} \left(\hat{H}_i^{\circ}\right)^{-1} \frac{d\mathcal{R}^{\circ}}{d\mu_k} + \bar{\kappa}^{-2} \left(\hat{H}_i^{\circ}\right)^{-1} \left(\sum_{lm} \frac{\partial \mathcal{R}_{(1)}}{\partial \alpha_{lm,(1)}} \frac{d\alpha_{lm,(1)}}{d\mu_k} + \frac{\partial \mathcal{R}_{(1)}}{\partial \mu_k}\right) + O\left(\bar{\kappa}^{-3}\right),$$

where, as usual, the partial derivatives imply that the production network is kept constant. From

(A.27), we have

$$\frac{d\mathcal{R}^\circ}{d\mu_k} = -\mathcal{L}^\circ \mathbf{1}_k,$$

and from (A.28), we have

$$\frac{\partial \mathcal{R}_{(1)}}{\partial \mu_k} = -\mathcal{L}_{(1)} \mathbf{1}_k.$$

Similarly, we use (A.28) to compute the partial derivative

$$\frac{\partial \mathcal{R}_{(1)}}{\partial \alpha_{ij,(1)}} = -\mathcal{L}^\circ \mathbf{1}_i \mathbf{1}_j^\top \mathcal{L}^\circ \mu - \mathcal{L}^\circ \mathbf{1}_i \mathbf{1}_j^\top \hat{H}_i^\circ \alpha_{i,(1)} + (\rho - 1)\, \mathcal{L}^\circ \Sigma \left(\mathcal{L}^\circ \mathbf{1}_i \mathbf{1}_j^\top \mathcal{L}^\circ \right)^\top \beta + (\rho - 1) \left(\mathcal{L}^\circ \mathbf{1}_i \mathbf{1}_j^\top \mathcal{L}^\circ \right) \Sigma \left[\mathcal{L}^\circ \right]^\top \beta \tag{A.39}$$

$$= \mathcal{L}^\circ \mathbf{1}_i \mathbf{1}_j^\top \left(-\mathcal{L}^\circ \mu + (\rho - 1)\, \mathcal{L}^\circ \Sigma \left[\mathcal{L}^\circ \right]^\top \beta - \hat{H}_i^\circ \alpha_{i,(1)} \right) + (\rho - 1)\, \mathcal{L}^\circ \Sigma \left(\mathcal{L}^\circ \mathbf{1}_i \mathbf{1}_j^\top \mathcal{L}^\circ \right)^\top \beta$$

$$= \mathcal{L}^\circ \mathbf{1}_i \mathbf{1}_j^\top \left(\mathcal{R}_{(0)} - \hat{H}_i^\circ \alpha_{i,(1)} \right) + (\rho - 1)\, \mathcal{L}^\circ \Sigma \left(\mathcal{L}^\circ \mathbf{1}_i \mathbf{1}_j^\top \mathcal{L}^\circ \right)^\top \beta.$$

From Proposition 9, $\alpha_{i,(1)} = \left(\hat{H}_i^\circ \right)^{-1} \mathcal{R}_{(0)}$, hence the first term in the last line of the expression above is zero. Finally, from (A.29) we find

$$\frac{d\alpha_{lm,(1)}}{d\mu_k} = -\mathbf{1}_m^\top \left(\hat{H}_l^\circ \right)^{-1} \mathcal{L}^\circ \mathbf{1}_k.$$

Grouping terms completes the proof. □

Proof of Corollary 13

Corollary 13. *The impact of an increase in Σ_{op} on the network is given by*

$$\frac{d\alpha_i}{d\Sigma_{op}} = \bar{\kappa}^{-1} \left(\hat{H}_i^\circ \right)^{-1} \underbrace{\left(\frac{\partial \mathcal{R}_{(0)}}{\partial \Sigma_{op}} + \bar{\kappa}^{-1} \frac{\partial \mathcal{R}_{(1)}}{\partial \Sigma_{op}} \right)}_{\text{Response of } \mathcal{R} \text{ with fixed network}} + \bar{\kappa}^{-2} \left(\hat{H}_i^\circ \right)^{-1} \underbrace{\left(\sum_{lm} \frac{d\alpha_{lm,(1)}}{d\Sigma_{op}} \times \frac{\partial \mathcal{R}_{(1)}}{\partial \alpha_{op,(1)}} \right)}_{\text{Impact of the network on } \mathcal{R}} + O\left(\bar{\kappa}^{-3} \right), \tag{A.35}$$

where the impact of Σ_{op} on $\mathcal{R}$ taking the network fixed is given by

$$\frac{\partial \mathcal{R}_{(0)}}{\partial \Sigma_{op}} + \bar{\kappa}^{-1} \frac{\partial \mathcal{R}_{(1)}}{\partial \Sigma_{op}} = (\rho - 1)\, \mathcal{L}^\circ \left[\frac{1}{2} \left(\mathbf{1}_o \mathbf{1}_p^\top + \mathbf{1}_p \mathbf{1}_o^\top \right) \right] \omega^\circ$$
$$+ \bar{\kappa}^{-1} \left[(\rho - 1)\, \mathcal{L}^\circ \left[\frac{1}{2} \left(\mathbf{1}_o \mathbf{1}_p^\top + \mathbf{1}_p \mathbf{1}_o^\top \right) \right] \mathcal{L}_{(1)}^\top \beta + (\rho - 1)\, \mathcal{L}_{(1)} \left[\frac{1}{2} \left(\mathbf{1}_o \mathbf{1}_p^\top + \mathbf{1}_p \mathbf{1}_o^\top \right) \right] \left(\mathcal{L}^\circ \right)^\top \beta \right],$$

and the change in $\mathcal{R}$ through the response of the network is given by

$$\frac{d\alpha_{lm,(1)}}{d\Sigma_{op}} \times \frac{\partial \mathcal{R}_{(1)}}{\partial \alpha_{lm,(1)}} = (\rho - 1) \left(\hat{H}_i^\circ \right)^{-1} \mathcal{L}^\circ \left[\frac{1}{2} \left(\mathbf{1}_o \mathbf{1}_p^\top + \mathbf{1}_p \mathbf{1}_o^\top \right) \right] \omega^\circ \times (\rho - 1)\, \mathcal{L}^\circ \Sigma \left(\mathcal{L}^\circ \mathbf{1}_l \mathbf{1}_m^\top \mathcal{L}^\circ \right)^\top \beta. \tag{A.36}$$

Proof. Differentiating (A.23) with respect to Σ_{op} yields

$$\begin{aligned}\frac{d\alpha_i}{d\Sigma_{op}} &= \bar{\kappa}^{-1}\frac{d\alpha_{i,(1)}}{d\Sigma_{op}} + \bar{\kappa}^{-2}\frac{d\alpha_{i,(2)}}{d\Sigma_{op}} + O\left(\bar{\kappa}^{-3}\right)\\ &= \bar{\kappa}^{-1}\left(\hat{H}_i^{\circ}\right)^{-1}\frac{d\mathcal{R}^{\circ}}{d\Sigma_{op}} + \bar{\kappa}^{-2}\left(\hat{H}_i^{\circ}\right)^{-1}\left(\sum_{lm}\frac{\partial\mathcal{R}_{(1)}}{\partial\alpha_{lm,(1)}}\frac{d\alpha_{lm,(1)}}{d\Sigma_{op}} + \frac{\partial\mathcal{R}_{(1)}}{\partial\Sigma_{op}}\right) + O\left(\bar{\kappa}^{-3}\right),\end{aligned}$$

where we have used the expressions for the first and second-order terms given by the Proposition 9. From (A.27), we have

$$\frac{d\mathcal{R}^{\circ}}{d\Sigma_{op}} = (\rho-1)\,\mathcal{L}^{\circ}\left[\frac{1}{2}\left(\mathbf{1}_o\mathbf{1}_p^{\top} + \mathbf{1}_p\mathbf{1}_o^{\top}\right)\right]\omega^{\circ},$$

and from (A.28), we have

$$\frac{\partial\mathcal{R}_{(1)}}{\partial\Sigma_{op}} = (\rho-1)\,\mathcal{L}^{\circ}\left[\frac{1}{2}\left(\mathbf{1}_o\mathbf{1}_p^{\top} + \mathbf{1}_p\mathbf{1}_o^{\top}\right)\right]\mathcal{L}_{(1)}^{\top}\beta + (\rho-1)\,\mathcal{L}_{(1)}\left[\frac{1}{2}\left(\mathbf{1}_o\mathbf{1}_p^{\top} + \mathbf{1}_p\mathbf{1}_o^{\top}\right)\right]\left[\mathcal{L}^{\circ}\right]^{\top}\beta.$$

From (A.39), we know that

$$\frac{\partial\mathcal{R}_{(1)}}{\partial\alpha_{ij,(1)}} = (\rho-1)\,\mathcal{L}^{\circ}\Sigma\left(\mathcal{L}^{\circ}\mathbf{1}_i\mathbf{1}_j^{\top}\mathcal{L}^{\circ}\right)^{\top}\beta.$$

Finally, from (A.29), we can compute

$$\frac{d\alpha_{lm,(1)}}{d\Sigma_{op}} = \left(\hat{H}_i^{\circ}\right)^{-1}\frac{d\mathcal{R}^{\circ}}{d\Sigma_{op}} = (\rho-1)\left(\hat{H}_i^{\circ}\right)^{-1}\mathcal{L}^{\circ}\left[\frac{1}{2}\left(\mathbf{1}_o\mathbf{1}_p^{\top} + \mathbf{1}_p\mathbf{1}_o^{\top}\right)\right]\omega^{\circ}.$$

□

H Additional information about the calibrated economy

We provide here additional information about the calibrated economy.

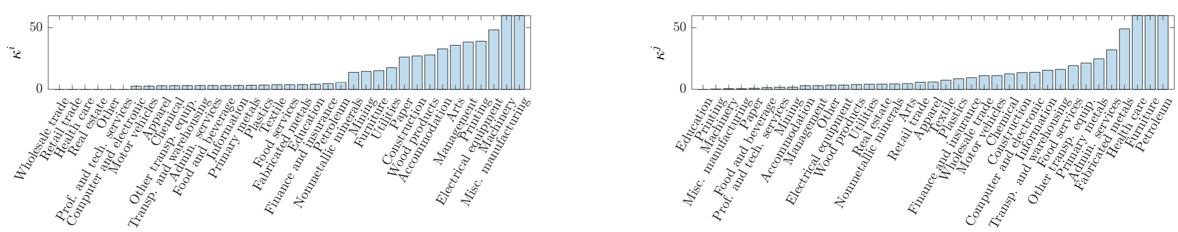
H.1 Cost of deviating from the ideal input shares

The overall mean of the elements of the calibrated cost matrix $\hat{\kappa}$ is 194 with a standard deviation of 447. The average and the standard deviation of the elements of the estimated vector $\hat{\kappa}^i$ are 14.11 and 17.56, respectively. The analogous statistics for $\hat{\kappa}^j$ are 13.73 and 17.18. To interpret these numbers, it helps to transform them into what they imply for productivity. If we increase one input share from its ideal value by one standard deviation in one sector, the average TFP loss for that sector is 0.06%.

To better understand the structure of the $\hat{\kappa}$ matrix, Figure 2 shows for each sector the elements of the vectors $\hat{\kappa}^i$ and $\hat{\kappa}^j$. As we can see, the amount of variation across sectors is quite substantial. The sectors with the highest $\hat{\kappa}^i$ are "Misc. manufacturing" and "Machinery", indicating that it is particularly costly for these sectors to deviate from their ideal input shares. The sectors with the highest $\hat{\kappa}^j$ are "Petroleum", "Furniture" and "Health care", implying that all firms tend to find it costly to adjust their input share of these sectors.

Figure 2. The calibrated costs of deviating from the ideal input shares

(a) Vector of costs κ^i (b) Vector of costs κ^j



H.2 Sectoral total factor productivity

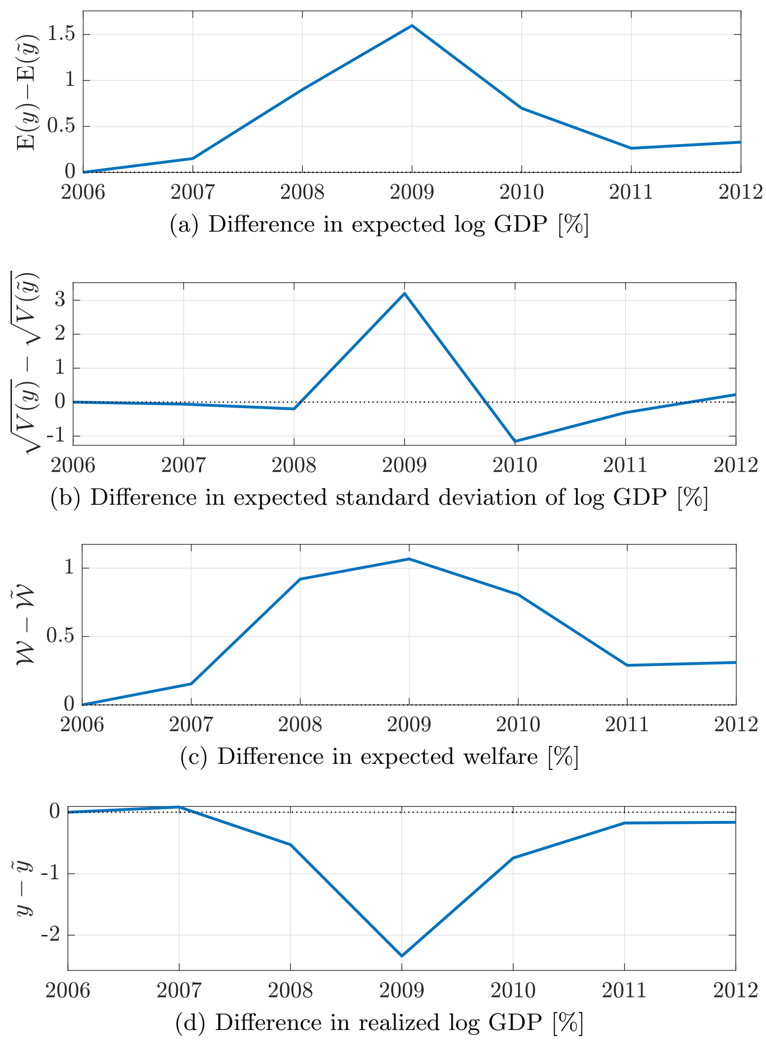
The estimated drift vector $\hat{\gamma}$ features substantial variation across sectors, indicating sizable dispersion in the trajectory of sectoral TFP. "Computer and electronic manufacturing" has the highest average annual growth in the sample, with ε_{it} growing 5.6% faster than the average sector. At the other end of the spectrum, productivity in "Food services" shrank by -2.9% per year relative to the average sector.

Similarly, the estimated covariance matrix $\hat{\Sigma}_t$ suggests that there is also substantial dispersion in uncertainty across sectors. The most volatile productivity is found in "Electrical equipment" with an average $\sqrt{\hat{\Sigma}_{iit}}$ of 38.0%, and the least volatile sector is "Real estate" with an average $\sqrt{\hat{\Sigma}_{iit}}$ of 1.8%. There is also a lot of variation across sectors in how much volatility changes over time. The standard deviation of $\sqrt{\hat{\Sigma}_{iit}}$ is largest for the "Electrical equipment" sector at 25.6% and smallest for "Real estate" at 1.1%.

H.3 Great Recession: Flexible vs fixed network

In this section, we explore the role of network flexibility during the Great Recession—the period in which the economy was hit by large adverse shocks (see Figure 5 in the main text). Specifically, we fix the network α at its 2006 pre-recession level and then hit the economy with the same shocks as in the baseline economy with endogenous network. Figure 3 shows how the baseline economy compares to the fixed-network alternative (denoted with tildes in the figure) over the years 2006 to 2012. We find that expected GDP (top panel) is higher under the flexible network. This is because firms are able to respond to changes in TFP and move away from sectors that are expected to perform badly. When doing so, firms become exposed to more productive but also more volatile suppliers, which results in an increase in GDP volatility (second panel). However, the first effect dominates, and welfare is quite substantially higher when the network is allowed to adjust (third panel). Interestingly, the economy with a flexible network does substantially worse in terms of realized GDP (bottom panel) during the Great Recession years. As evident from the two top panels, firms optimally choose to be exposed to more productive but riskier suppliers. During the Great Recession, some of those risks were realized, pushing realized GDP down for the baseline case.

Figure 3. The role of network flexibility during the Great Recession



Notes: The differences between the series implied by the full model (without tildes) and the model in which the network is fixed at its 2006 level (with tildes). All differences are expressed in percentage terms.

H.4 Input shares and Domar weights

In this appendix, we compare the behavior of the input shares and the Domar weights in the baseline economy and in the two alternative economies introduced in Section 8: the one with $\Sigma_t = 0$ and the economy in which production techniques are chosen after observing ε_t. As we can see from Table I all versions of the model perform almost identically in terms of average shares and Domar weights. The standard deviations differ however across models. Specifically, the baseline model, in which firms care about uncertainty, features standard deviations that 4% to 8% lower than in the alternatives, depending on the precise comparison.

TABLE I. Domar weights and input shares in the model and in the data

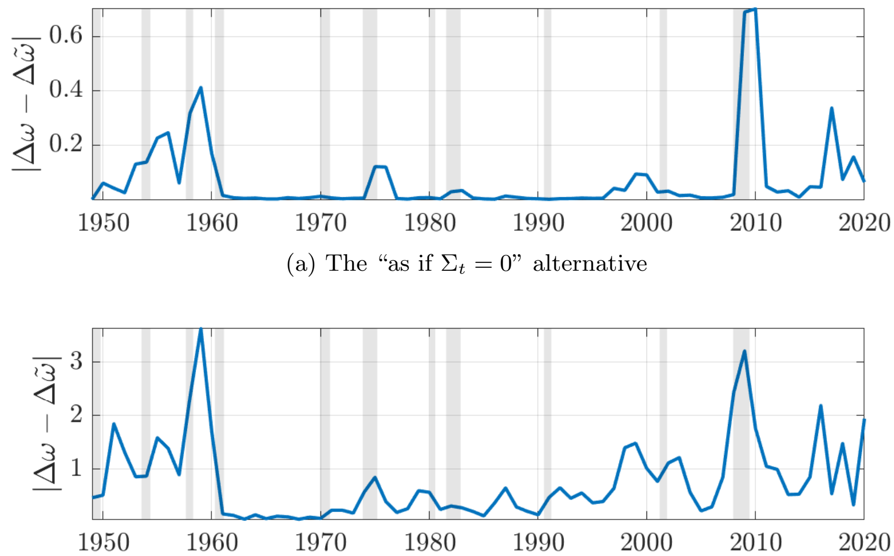
	Statistic	Version of the model			
		Data	Baseline	$\Sigma_t = 0$	Known ε_t
(1)	Average Domar weight $\bar{\omega}_j$	0.047	0.032	0.032	0.032
(2)	Standard deviation $\sigma(\omega_j)$	0.0050	0.0021	0.0022	0.0023
(3)	Coefficient of variation $\sigma(\omega_j)/\bar{\omega}_j$	0.107	0.066	0.070	0.070
(4)	Average share, $\bar{\alpha}_{ij}$	0.013	0.008	0.008	0.008
(5)	Standard derivation $\sigma(\alpha_{ij})$	0.0048	0.0023	0.0024	0.0024
(6)	Coefficient of variation $\sigma(\alpha_{ij})/\bar{\alpha}_{ij}$	0.37	0.30	0.31	0.31

Notes: For each sector, we compute the time series of its Domar weight ω_{jt}, as well as their mean $\bar{\omega}_j$ and standard deviation $\sigma(\omega_j)$. Rows (1) and (2) report the cross-sectional average of these statistics. Row (3) is the ratio of rows (2) and (1). For each pair of sectors, we compute the time series of the input share α_{ijt}, as well as their their mean $\bar{\alpha}_{ij}$ and standard deviation $\sigma(\alpha_{ij})$. Rows (4) and (5) report the cross-sectional average of these statistics. Row (6) is the ratio of rows (5) and (4). The "Baseline" model is the model in which risk-averse firms choose techniques before TFP shocks ε are realized. The "$\Sigma_t = 0$" model is the model in which the planner selects the network as if $\Sigma = 0$. The "Known ε_t" model is the model in which firms choose techniques after the TFP shocks ε are realized.

The response of the network to uncertainty differs particularly strongly across models during periods of high uncertainty. To show this, we compute changes in sectoral Domar weights $\Delta\omega_{it} = \omega_{it} - \omega_{i,t-1}$ in the baseline model and in the two alternatives. As usual, we denote changes in sectoral Domar weights in the alternative models by tildes, i.e. $\Delta\tilde{\omega}_{it}$. We then compute the cross-sectional average of the absolute differences between $\Delta\omega_{it}$ and $\Delta\tilde{\omega}_{it}$, and normalize it by the cross-sectional average of standard deviations of $|\Delta\omega_{it}|$. This measure captures the difference between models in how Domar weights change over time.

Figure 4 shows the resulting graphs. In the top panel, the "$\Sigma_t = 0$" model is used as alternative. In the bottom panel, the "known ε_t" model is used as alternative. In the top panel the differences are particularly pronounced during high-uncertainty episodes, when risk-averse firms actively switch to safer production inputs. In the bottom panel, the Domar weights deviate from the baseline model much more. This is because the production network adapts to the specific ε_t draw, and so the differences are visible even in relatively tranquil times.

Figure 4. Average of absolute differences in Domar weight growths in the postwar period



(a) The "as if $\Sigma_t = 0$" alternative

(b) The "known ε_t" alternative

Notes: Panel (a): difference between the series implied by the baseline model (without tildes) and the "as if $\Sigma_t = 0$" alternative (with tildes); Panel (b): difference between the baseline model (without tildes) and the alternative in which firms choose techniques after TFP shocks ε are realized (with tildes). Both economies are hit by the same shocks that are filtered out from the TFP data under our baseline model. Both series are normalized by the cross-sectional average of the standard deviations of growths in sectoral Domar weights.

H.5 Robustness exercises

In this appendix, we provide two robustness exercises. First, we investigate what happens with the coefficient of risk aversion varies. Second, we explore how the calibrated economy behaves when β can change every period.

Sensitivity to the risk aversion parameter ρ

In this section, we investigate the sensitivity of our results to the value of the risk aversion parameter ρ. To do so, we solve the model for different values of ρ without recalibrating the matrix κ. We then compare this economy to the alternative with $\Sigma = 0$. Not surprisingly, we find that ignoring uncertainty is costlier for higher values of ρ (Table II).

The economy also responds to the spike in uncertainty during the Great Recession much more for $\rho = 10$ (Figure 5). Specifically, if $\rho = 10$, the network adjusts such that the standard deviation of GDP is almost 4.2% lower in 2009 relative to the risk-neutral alternative (second panel, yellow crossed line). Although this adjustment is associated with a sizable decline in expected GDP (-0.8%; first panel), welfare raises substantially (1.8%; third panel). This is because the representative household enjoys a larger utility gain from a reduction in uncertainty under a higher risk aversion parameter.

TABLE II. Uncertainty, GDP and welfare: the Role of risk aversion

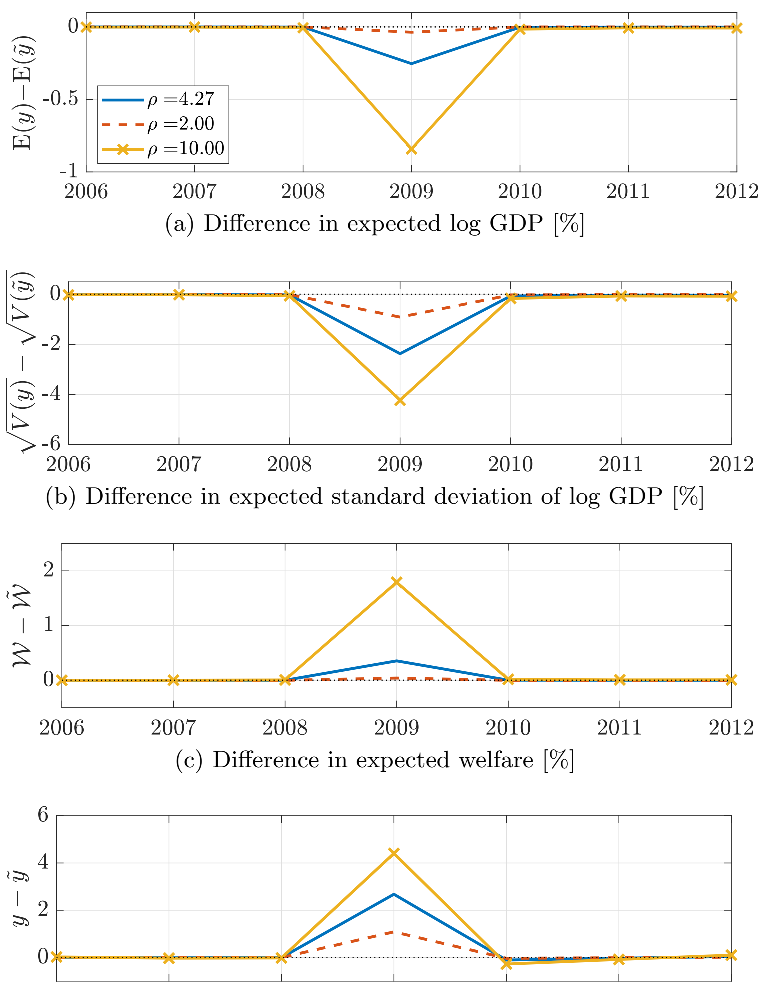
	Comparison with $\Sigma = 0$ model		
	$\rho = 2$	$\rho = 4.27$ (baseline)	$\rho = 10$
Expected log GDP $\mathrm{E}[y(\alpha)]$	+0.001%	+0.008%	+0.033%
Std. dev. of log GDP $\sqrt{\mathrm{V}[y(\alpha)]}$	+0.038%	+0.105%	+0.208%
Welfare $\mathcal{W}$	−0.001%	−0.010%	−0.057%

Notes: Baseline economy variables minus their counterparts in the $\Sigma = 0$ alternative for different values of risk aversion ρ.

Time-varying consumption shares

In this appendix, we consider a version of the calibrated economy in which we let the β preference vector change over time to match the observed consumption shares in the data. Figure 6 shows the difference between that economy and the $\Sigma = 0$ alternative in which uncertainty has no impact on the production network. As we can see, this figure is quite similar to Figure 4 (left column) in the main text, suggesting that allowing β to change over time does not have a large effect on the impact of uncertainty on the network.

Figure 5. The role of uncertainty during the Great Recession: Role of risk aversion



(a) Difference in expected log GDP [%]

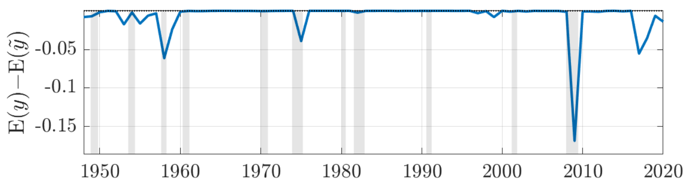
(b) Difference in expected standard deviation of log GDP [%]

(c) Difference in expected welfare [%]

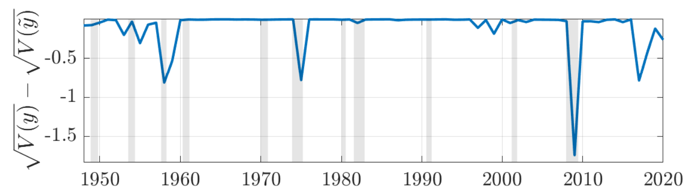
(d) Difference in realized log GDP [%]

Notes: The differences between the series implied by the models featuring various degrees of risk aversion (without tildes) and the "as if $\Sigma_t = 0$" alternative (with tildes). All differences are expressed in percentage terms.

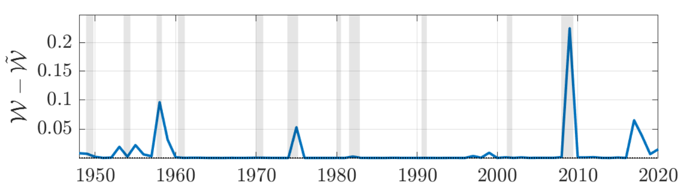
Figure 6. The role of uncertainty in the postwar period with time-varying β



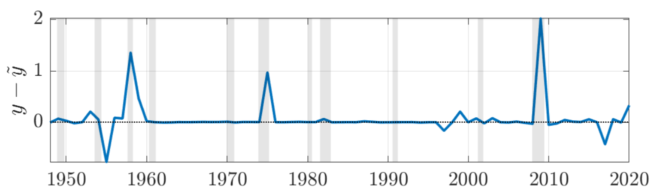
(a) Difference in expected log GDP [%]



(b) Difference in expected standard deviation of log GDP [%]



(c) Difference in expected welfare [%]



(d) Difference in realized log GDP [%]

Notes: The differences between the series implied by the baseline model with time-varying β (without tildes) and the "as if $\Sigma_t = 0$" alternative (with tildes). Both economies are hit by the same shocks that are filtered out from the TFP data under our baseline model. All differences are expressed in percentage terms.

I Wedges and inefficient allocation

In this appendix, we consider a version of the competitive economy of Section 2 with wedges. To do so, we modify our setup along the lines of Acemoglu and Azar (2020). Specifically, we assume that firms in industry i sell their goods at a markup $\tau_i \geq 0$ over their unit cost. A fraction $\zeta_i \in [0,1]$ of the revenue from the distortions is rebated to the representative household. The remaining $1-\zeta_i$ share is pure waste. We assume that τ_i and ζ_i are exogenous and do not depend on ε. Below, we first describe the decentralized equilibrium with wedges and then show that there exists a distorted "planner" whose decisions coincide with the distorted equilibrium. Finally, we characterize how distortions affect equilibrium outcomes.

I.1 A distorted equilibrium

Several parts of the model are not affected by the wedges. In particular, the objective function of the household remains unchanged. Its budget constraint must however be adjusted to take into account the profits generated by the wedges. It becomes

$$\sum_{i=1}^{n} P_i C_i \leq 1 + T(\alpha),$$

where $T(\alpha) = \sum_{i=1}^{n} \zeta_i \frac{\tau_i}{1+\tau_i} P_i Q_i$ is the rebate due to distortions. As we show below, T depends on α but not on ε, which justifies the notation $T(\alpha)$. In the absence of distortions ($\tau_i = 0$ for all i) or if distortions are pure waste ($\zeta_i = 0$ for all i), $T = 0$. The additional term in the budget constraint implies a different stochastic discount factor Λ. From the first-order conditions of the household it follows that

$$\lambda = (\rho - 1)\,\beta^{\top} p - \rho \log(1 + T), \tag{A.40}$$

where $\lambda = \log \Lambda$ is the log of the stochastic discount factor.

On the side of the firm, the cost minimization problem (7) is unaffected by the wedges, and so the unit cost K_i conditional on a technique α_i and a price vector P can still be written as (8). Similarly, the technique choice problem of the firm, conditional on prices, is unaffected and is still defined by (9). We will see however that in equilibrium the wedges affect the technique choices of the firms through their impact on prices.

Equilibrium conditions

We now turn to the market clearing conditions and the pricing equations. Those are affected by the wedges. In particular, the pricing equation (10) becomes

$$P_i = (1+\tau_i)\, K_i(\alpha, P), \tag{A.41}$$

such that prices are set at a markup over unit cost. We can combine this equation with (8) to write an expression for log prices as a function of the network α,

$$p = -\mathcal{L}(\alpha)\left(\varepsilon + a(\alpha) - \log(1+\tau)\right), \tag{A.42}$$

where $\log(1+\tau)$ is a column vector with typical element $\log(1+\tau_i)$. As we can see, wedges τ affect prices as productivity shifters.

The market clearing condition (11) for good i must also be adjusted for the potential loss in resources. It becomes

$$Q_i\left(1-(1-\zeta_i)\frac{\tau_i}{1+\tau_i}\right) = C_i + \sum_j X_{ji}. \tag{A.43}$$

We can use these equations to find an expression for the rebate to the household T. Combining the first-order conditions of the firms with (A.41) and (A.43), we get

$$T(\alpha) = \begin{pmatrix} \zeta_1\tau_1 \\ \zeta_2\tau_2 \\ \vdots \\ \zeta_n\tau_n \end{pmatrix}^{\top} \left[\begin{pmatrix} 1+\zeta_1\tau_1 & 0 & \dots & 0 \\ 0 & 1+\zeta_2\tau_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1+\zeta_n\tau_n \end{pmatrix} - \beta \begin{pmatrix} \zeta_1\tau_1 \\ \zeta_2\tau_2 \\ \vdots \\ \zeta_n\tau_n \end{pmatrix}^{\top} - \alpha^{\top} \right]^{-1} \beta, \tag{A.44}$$

and so we can fully characterize the stochastic discount factor (A.40) for a given network α. Note also that $T = 0$ whenever $\zeta = 0$ or $\tau = 0$.

Using the expression for T together with the price vector p, we can write log GDP as

$$y = \beta^{\top}\mathcal{L}(\alpha)\left(\varepsilon + a(\alpha) - \log(1+\tau)\right) + \log\left(1 + T(\alpha, \zeta, \tau)\right). \tag{A.45}$$

We see that the wedges have two different impacts on GDP. First, the distortions effectively lead to a decline in productivity through $\log(1+\tau)$. At the same time, the part of these distortions that is rebated to the household has a positive impact on GDP through T.

Finally, in the following lemma we characterize the comparative statics of the rebate amount.

Lemma 13. *Holding everything else equal, $T(\alpha)$ increases in α_{ij} for all i, j.*

Proof. Denote $\chi_i = \zeta_i\tau_i$. Rewrite (A.44) as

$$T(\alpha,\chi)=\chi^{\top}\left[I+\operatorname{diag}(\chi)-\beta\chi^{\top}-\alpha^{\top}\right]^{-1}\beta,$$

where $\operatorname{diag}(\chi)$ is a diagonal matrix with the vector χ on its main diagonal. Note that $I+\operatorname{diag}(\chi)-\chi\beta^{\top}-\alpha$ is a diagonally dominant Z-matrix with a positive main diagonal. Therefore, it is an M-matrix whose inverse has all nonnegative inputs. Therefore, $\left[I+\operatorname{diag}(\chi)-\beta\chi^{\top}-\alpha^{\top}\right]^{-1}$ is a nonnegative matrix as well. Differentiating $T(\alpha,\chi)$ with respect to α_{ij} yields

$$\frac{\partial T}{\partial \alpha_{ij}}=\chi^{\top}\left[I+\operatorname{diag}(\chi)-\beta\chi^{\top}-\alpha^{\top}\right]^{-1}\left(\mathbf{1}_j\mathbf{1}_i^{\top}\right)\left[I+\operatorname{diag}(\chi)-\beta\chi^{\top}-\alpha^{\top}\right]^{-1}\beta.$$

Since all vectors and matrices in the right-hand side of the above expression are nonnegative, $\frac{\partial T}{\partial \alpha_{ij}}\geq 0$. □

Firm decision

In the presence of wedges, the technique choice by firms is described in the following lemma. It is a direct analogue of Lemma 2 from the main text. The only difference is that expected equilibrium log prices $p^*(\alpha)$ in A.47 needs to be adjusted by $\mathcal{L}(\alpha^*)\log(1+\tau)$, as follows from Equation (A.42).

Lemma 14. *In the presence of wedges, the equilibrium technique choice problem of the representative firm in sector i is*

$$\alpha_i^*\in\arg\max_{\alpha_i\in\mathcal{A}_i} a_i(\alpha_i)-\sum_{j=1}^{n}\alpha_{ij}\mathcal{R}_j(\alpha^*),\tag{A.46}$$

where

$$\mathcal{R}(\alpha^*)=\mathrm{E}\left[p(\alpha^*)\right]+\mathrm{Cov}\left[p(\alpha^*),\lambda(\alpha^*)\right],\tag{A.47}$$

is the equilibrium risk-adjusted price of good j, and where

$$\mathrm{E}\left[p(\alpha^*)\right]=-\mathcal{L}(\alpha^*)\left(\mu-\log(1+\tau)+a(\alpha^*)\right)\ \textit{and}\ \mathrm{Cov}\left[p(\alpha^*),\lambda(\alpha^*)\right]=(\rho-1)\,\mathcal{L}(\alpha^*)\,\Sigma\left[\mathcal{L}(\alpha^*)\right]^{\top}\beta.$$

Proof. The proof is analogous to that of Lemma 2, with log prices given by (A.42) instead of (12). □

I.2 A distorted planner's problem

In the main text, we exploit the fact that the equilibrium allocation can be written as the outcome of the planner's optimization problem. Here, because of the distortions, it is no longer true that the equilibrium coincides with the planner's allocation. We can however derive the problem of a *distorted* fictitious planner whose preferred allocation coincides with the distorted equilibrium. We can then take advantage of that optimization problem to characterize the distorted equilibrium.

We define this distorted planner's problem as

$$\mathcal{W}^d \equiv \max_{\alpha\in\mathcal{A}} \mathrm{E}\left[y^d(\alpha)\right] - \frac{1}{2}(\rho-1)\,\mathrm{V}\left[y^d(\alpha)\right], \tag{A.48}$$

where

$$y^d(\alpha) = \beta^\top \mathcal{L}(\alpha)\left(\varepsilon + a(\alpha) - \log(1+\tau)\right)$$

is what log GDP *would be* if nothing was rebated to the household.

We can rewrite the distorted planner's problem in terms of Domar weights.

$$\mathcal{W}^d \equiv \max_{\omega\in\mathcal{O}} \omega^\top\left(\mu - \log(1+\tau)\right) + \bar{a}(\omega) - \frac{1}{2}(\rho-1)\,\omega^\top\Sigma\omega, \tag{A.49}$$

where

$$\begin{aligned} \bar{a}(\omega) = &\max_{\alpha\in\mathcal{A}} \omega^\top a(\alpha) \\ &\text{s.t.}\, \omega^\top = \beta^\top \mathcal{L}(\alpha). \end{aligned} \tag{A.50}$$

Similar to the main model, the distorted planner's problem has a unique solution because the distorted planner's objective function is concave in ω, and there exists unique $\alpha = \alpha(\omega)$ solving (A.50). Furthermore, following the same steps as in the proof of Proposition 1, we can find that any solution to the firm's problem is also a solution to the distorted planner's problem.

Proposition 10. *There exists a unique equilibrium.*

Proof. The proof is analogous to that of Proposition 1. □

The unique equilibrium is no longer efficient because the distorted planner does not maximize the true welfare of the representative household. Specifically, the distorted planner does not take into account the fact that part of the distortion income is rebated to the household.

I.3 Characterizing the distorted equilibrium

We can use the distorted planner's problem to characterize the distorted equilibrium. The next proposition describes how the Domar weights are affected by beliefs (μ, Σ) and the wedges τ.

Proposition 11. *The Domar weight ω_i of sector i is increasing in μ_i, decreasing in Σ_{ii} and decreasing in τ_i.*

Proof. The proof is similar to that of Proposition 2. Recall that

$$\mathcal{W}^d = \max_{\omega \in \mathcal{O}} \omega^\top (\mu - \log(1+\tau)) + \bar{a}(\omega) - \frac{1}{2}(\rho - 1)\omega^\top \Sigma \omega.$$

By the envelope theorem,

$$\frac{d\mathcal{W}^d}{d\mu_i} = -\frac{d\mathcal{W}^d}{d\log(1+\tau_i)} = \omega_i \text{ and } \frac{d\mathcal{W}^d}{d\Sigma_{ij}} = -\frac{1}{2}(\rho-1)\omega_i\omega_j.$$

With these derivatives in hand, we can follow analogous steps to those in the proof of Proposition 2 to find that an increase in μ_i or a decline in Σ_{ii} leads to a higher ω_i. Finally, an increase in τ_i is equivalent to a decline in μ_i, and so the last part of the proposition follows. ☐

Next, we investigate how changes in beliefs and wedges affect welfare, which can be written as

$$\mathcal{W}(\alpha^*) = \mathcal{W}^d(\alpha^*) + \log(1 + T(\alpha^*)),$$

where $\alpha^* = \alpha^*(\omega^*)$ solves (A.50) and ω^* solves (A.49).

Proposition 12. *In the distorted equilibrium, the following holds.*

1. *The impact of an increase in μ_i on welfare is given by*

$$\frac{d\mathcal{W}}{d\mu_i} = \omega_i + \frac{1}{1+T(\alpha^*)}\sum_{k,l}\frac{\partial T}{\partial \alpha_{kl}}\frac{d\alpha^*_{kl}}{d\mu_i}.$$

2. *The impact of an increase in Σ_{ij} on welfare is given by*

$$\frac{d\mathcal{W}}{d\Sigma_{ij}} = -\frac{1}{2}(\rho-1)\omega_i\omega_j + \frac{1}{1+T(\alpha^*)}\sum_{k,l}\frac{\partial T}{\partial \alpha_{kl}}\frac{d\alpha^*_{kl}}{d\Sigma_{ij}}.$$

3. *If $\zeta = 0$, then the results of Proposition 4 hold in the distorted equilibrium.*

Proof. Taking derivative of $\mathcal{W}$ with respect to μ_i yields

$$\frac{d\mathcal{W}}{d\mu_i} = \frac{d\mathcal{W}^d(\alpha^*)}{d\mu_i} + \frac{1}{1+T(\alpha^*)}\sum_{k,l}\frac{\partial T}{\partial \alpha_{kl}}\frac{d\alpha^*_{kl}}{d\mu_i} = \omega_i + \frac{1}{1+T(\alpha^*)}\sum_{k,l}\frac{\partial T}{\partial \alpha_{kl}}\frac{d\alpha^*_{kl}}{d\mu_i}.$$

Similar steps yield the expression for the derivative with respect to Σ_{ij}. Note that $T = 0$ when $\zeta = 0$ and so $\frac{\partial T}{\partial \alpha_{kl}} = 0$ for all k, l in this case. We then get the expressions of Proposition 4. ☐

We see from these equations that changes in μ and Σ have two effects on welfare. There is a term that is the same as in the efficient allocation, as described by Proposition 4. But there is also a second term that reflects how changes in the production network lead to a higher or smaller rebate to the household. For example, if conditions of part 1 of Proposition 4 are satisfied, an increase in μ_i leads to an increase in all shares, that is, $\frac{d\alpha^*_{kl}}{d\mu_i} > 0$ for all k, l. This in turn increases the rebated amount T by Lemma 13 and, as a result, $\frac{d\mathcal{W}}{d\mu_i} > \omega_i$.

J Other forms of uncertainty

In this appendix, we discuss how our model can be extended to handle other types of uncertainty. We assume that when picking techniques, firms not only face uncertainty about the productivity shocks ε but are also uncertain about 1) household's preferences; 2) labor supply; 3) distortions. As in the baseline model, all uncertainties are realized before firms pick quantities and the household picks its consumption basket. We start by describing how these additional types of uncertainty can be introduced in our framework.

Household's preferences The household's demand for different good types is determined by the vector of preference parameters β. In this appendix, we assume that β is unknown when firms pick production techniques. We maintain the restrictions that $\beta_i > 0$ for all i and $\sum_{i=1}^{n} \beta_i = 1$. One natural distribution that satisfies these restrictions is the Dirichlet distribution.

Labor supply In the baseline model, the labor supply is fixed at one. In this appendix, we assume instead that the total labor supply is given by $\bar{L} > 0$ and that $\bar{L}$ is unknown when firms pick production techniques. Uncertainty in $\bar{L}$ might stem from, for example, changes in immigration, retirement, or health (e.g., stay-at-home orders or mandated shutdowns) regulations that affect the size of the labor force.

Distortions We introduce distortions in the same way as in Appendix I. Specifically, the price of good i exceeds the unit cost for firms, namely, $P_i = (1 + \tau_i) K_i (\alpha_i, P)$, where $K_i (\alpha, P)$ is given by (8). We assume that a fraction $1 - \zeta_i \in [0, 1]$ of the distortion revenue is pure waste. Note that these distortions can be driven by various sources, including government interventions and markups.[5] To study the role of distortion uncertainty, we assume that when firms pick their production techniques, they are uncertain about τ.

To focus on how different types of uncertainty affect equilibrium network, we are going to make the following assumption.

Assumption 2. *The shocks to productivity (ε), preferences (β), distortions (τ) and labor supply ($\bar{L}$) are independent.*

[5] Our modeling of distortions follows Acemoglu and Azar (2020). Liu (2019) considers a richer structure of government interventions. Importantly, these papers do not study the role of distortion uncertainty.

Analysis

After all uncertainties are realized, the representative household chooses its consumption basket and firms pick quantities of production factors given their production techniques α. Several parts of the model remain the same as in the main text. In particular, the objective function of the household remains unchanged. Its budget constraint must however be adjusted to

$$\sum_{i=1}^{n} P_i C_i \leq \bar{L} + T,$$

where the first term on the right-hand side is the household's labor income (recall that we can normalize wage to one without loss of generality), and the second term on the right-hand side $T = \sum_{i=1}^{n} \zeta_i \frac{\tau_i P_i Q_i}{1+\tau_i}$ is the amount of distortion revenue rebated to the household. A different form of the right-hand side of the budget constraint implies a different stochastic discount factor Λ. From the first-order conditions of the household we can write the log of the stochastic discount factor as

$$\lambda = (\rho - 1)\, \beta^{\top} p - \rho \log\left(\bar{L} + T\right). \tag{A.51}$$

On the firms' side, the cost minimization problem (7) is unaffected, and so the unit cost K_i conditional on a technique α_i and a price vector P can still be written as (8). Similarly, the technique choice problem of the firm, conditional on prices, is unaffected and is still defined by (9).

We now turn to the market clearing conditions and the pricing equations. Those are not the same as in our main model. In particular, the pricing equation (10) becomes

$$P_i = (1 + \tau_i)\, K_i(\alpha, P). \tag{A.52}$$

We can combine this equation with (8) to write an expression for log prices as a function of the network α,

$$p = -\mathcal{L}(\alpha)\left(\varepsilon + a(\alpha) - \log(1+\tau)\right), \tag{A.53}$$

where $\log(1+\tau)$ is a column vector with typical element $\log(1+\tau_i)$. As we can see, taxes τ affect prices as productivity shifters.

The market clearing condition (11) for good i must also be adjusted for the potential loss in resources. It becomes

$$Q_i \left(1 - (1 - \zeta_i) \frac{\tau_i}{1+\tau_i}\right) = C_i + \sum_j X_{ji}. \tag{A.54}$$

We can use these equations to find an expression for the rebate to the household T. Combining the first-order conditions of the firms with (A.52) and (A.54), we get

$$T = \bar{L} \sum_{i=1}^{n} \zeta_i \tau_i \hat{\omega}_i, \tag{A.55}$$

where

$$\hat{\omega} = \left[\begin{pmatrix} 1+\zeta_1\tau_1 & 0 & \dots & 0 \\ 0 & 1+\zeta_2\tau_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1+\zeta_n\tau_n \end{pmatrix} - \beta \begin{pmatrix} \zeta_1\tau_1 \\ \zeta_2\tau_2 \\ \vdots \\ \zeta_n\tau_n \end{pmatrix}^{\top} - \alpha^{\top} \right]^{-1} \beta \tag{A.56}$$

is a vector of sectoral weights in the total factor income, $\hat{\omega}_i = \frac{P_i Q_i}{W\bar{L}} = \frac{P_i Q_i}{\bar{L}}$. In the absence of taxes ($\tau = 0$) or if taxes are purely wasteful ($\zeta = 0$), the rebate amount is zero, $T = 0$, and $\hat{\omega} = \omega = \mathcal{L}^{\top}\beta$ is the Domar weight vector.

In the main model with only ε uncertainty, firms in sector i prefer techniques that are more productive, in the sense of increasing $a_i(\alpha_i)$, and have low risk-adjusted prices $\mathcal{R}_j$ (Lemma 2). The following lemma shows how this result can be extended in the model with multiple sources of uncertainty.

Lemma 15. *The technique choice of the representative firm in sector i solves*

$$\alpha_i^* \in \arg\min_{\alpha_i \in \mathcal{A}_i} -a_i(\alpha_i) + \sum_{j=1}^{n} \alpha_{ij} \hat{\mathcal{R}}_{ij}(\alpha^*), \tag{A.57}$$

where

$$\hat{\mathcal{R}}_{ij} = \mathrm{E}[p_j] + \frac{\mathrm{Cov}[p_j, \Lambda\hat{\omega}_i]}{\mathrm{E}[\Lambda\hat{\omega}_i]}. \tag{A.58}$$

Here Λ is the marginal utility of the representative household with respect to changes in income, and $\hat{\omega}_i$ is the weight of sector i in the total factor income.

Proof. The objective function of firm in sector i is given by (9). The log stochastic discount factor is (A.51), where T is given by (A.55); demand for sector i's good is $Q_i = \frac{\hat{\omega}_i \bar{L}}{P_i}$, where $\hat{\omega}$ is given by (A.56) and $\log P$ is given by (A.53); and the unit cost function is (8). Combining them together, the objective function (9) becomes

$$\alpha_i^* \in \arg\min_{\alpha_i \in \mathcal{A}_i} \mathrm{E} \exp \left(\underbrace{(\rho-1)\sum_i \beta_i p_i - \rho \log \bar{L} - \rho \log \left(1 + \sum_{i=1}^n \zeta_i \tau_i \hat{\omega}_i \right)}_{=\log \Lambda} \right.$$

$$\left. + \underbrace{\log \bar{L} + \log \hat{\omega}_i + \log(1+\tau_i) - p_i}_{=\log Q_i} \underbrace{- \varepsilon_i - a_i(\alpha_i) + \sum_j \alpha_{ij} p_j}_{=\log K_i} \right). \tag{A.59}$$

Taking the first-order condition with respect to α_{ij} and imposing that in equilibrium $\alpha = \alpha^*$, we find

$$\mathrm{E}\left[\Lambda \bar{L} \hat{\omega}_i \left(-\frac{\partial a(\alpha_i^*)}{\partial \alpha_{ij}} + p_j\right)\right] + \chi_{ij}^e - \gamma_i^e = 0. \tag{A.60}$$

Notice that we can write $\Lambda\left(\bar{L}\right) = \bar{L}^{-\rho}\Lambda\left(\bar{L}=1\right)$. Therefore, under Assumption 2, the equilibrium network is unaffected by $\bar{L}$. Then (A.60) can be rewritten as

$$\mathrm{E}\left[\Lambda \hat{\omega}_i\right]\left(-\frac{\partial a(\alpha_i^*)}{\partial \alpha_{ij}} + \mathrm{E}\left[p_j\right]\right) + \mathrm{Cov}\left[\Lambda \hat{\omega}_i, p_j\right] + \tilde{\chi}_{ij}^e - \tilde{\gamma}_i^e = 0, \tag{A.61}$$

where we redefine Lagrange multipliers by dividing them by $\mathrm{E}\,\bar{L}$. This is the first-order condition of (A.57) with respect to α_{ij}. □

From (8), a marginal change in α_{ij} changes the log unit cost of firms in sector i by $-\frac{\partial a_i}{\partial \alpha_{ij}} + p_j$. Firms care not only about this effect but also how it comoves with the household's marginal utility with respect to changes in income Λ, adjusted by the relative importance of sector i, i.e. $\hat{\omega}_i$. Under normality of the productivity shocks ε, (A.58) can be rewritten as[6]

$$\hat{\mathcal{R}}_{ij} = \mathrm{E}_{\beta,\tau}\left[\mathcal{R}_j\right] + \frac{\mathrm{Cov}_{\beta,\tau}\left[\mathcal{R}_j, \left(\mathrm{E}_\varepsilon \Lambda\right)\hat{\omega}_i\right]}{\mathrm{E}_{\beta,\tau}\left[\left(\mathrm{E}_\varepsilon \Lambda\right)\hat{\omega}_i\right]}, \tag{A.62}$$

where $\mathrm{E}_x(\cdot)$ means that we take expectations with respect to the random variable x, and where $\mathcal{R}$ is given by

$$\mathcal{R} = \underbrace{-\mathcal{L}(\alpha^*)\left(\mu + a(\alpha^*) - \log(1+\tau)\right)}_{\mathrm{E}_\varepsilon[p]} + \underbrace{(\rho-1)\,\mathcal{L}(\alpha^*)\,\Sigma\left[\mathcal{L}(\alpha^*)\right]^\top \beta}_{\mathrm{Cov}_\varepsilon[p,\lambda]},$$

which is a direct analogue of (18). When picking their production techniques, firms adjust their beliefs about suppliers' prices not only for risk in ε but also for other risk types, as captured by the

[6] In particular, we use the law of total covariance and the fact that if x_1 and x_2 are normal variables, $\mathrm{Cov}(x_1, X_2) = \mathrm{Cov}(x_1, x_2)\,\mathrm{E}(X_2)$, where $X_2 = \exp(x_2)$.

second term on the right-hand side of (A.62). Notably, the labor supply uncertainty and, in fact, the level of $\bar{L}$ do not affect firms' production technique choices. As shown in the proof of Lemma 15, $\bar{L}$ enters the firm's objective (8) as a scaling factor of sectoral output Q_i and stochastic discount factor Λ. Therefore, under Assumption 2, the labor supply $\bar{L}$ does not affect firms' technique choices, and thus can be normalized to 1 without loss of generality.

Proposition 13. *The following statements about the impacts of different types of uncertainty on equilibrium network hold.*

1. *Labor supply uncertainty does not matter for the production network;*
2. *If distortions are purely wasteful, $\zeta = 0$, then uncertainty about $\log(1+\tau)$ has an equivalent impact on the production network as uncertainty about ε;*
3. *Uncertainty about the household's preferences do not matter for the production network if ε and τ are nonrandom.*

Proof. Part 1 has been already established in the proof of Lemma 15. To prove part 2, impose $\zeta = 0$ and denote $\hat{\varepsilon} = \varepsilon - \log(1+\tau)$. Then $\hat{\omega}$ does not explicitly depend on τ and ε, and the price function (A.53) and the firm's objective function (A.59) depend only on $\hat{\varepsilon}$, not on ε and τ separately. Finally, part 3 follows from the fact that the price (A.53) does not depend on β. Therefore, if ε and τ are nonrandom, (A.61) can be written as

$$-\frac{\partial a(\alpha_i^*)}{\partial \alpha_{ij}} + p_j + \check{\chi}_{ij}^e - \check{\gamma}_i^e = 0,$$

where we redefine Lagrange multipliers by dividing them by $\mathrm{E}[\Lambda \hat{\omega}_i]$. It is then clear that α^* does not depend on β. □

Proposition 13 justifies our focus on uncertainty in productivity ε in the main text. First, as discussed above, labor supply uncertainty does not matter for the production network. Second, shocks to productivity and to taxes have analogous effects on firms' decisions. In particular, a high distortion τ_i makes sector i an expensive supplier, which for other firms is equivalent to firms in industry i being unproductive. Notably, shocks to ε and to $\log(1+\tau)$ are no longer equivalent if distortions are not purely wasteful. In that case, some of the distortion revenues are rebated to the household, affecting the stochastic discount factor. Finally, preference uncertainty matters *only if* there is uncertainty about ε or τ. To understand this result, notice that the price vector (A.53) does not depend on the household's preferences because it is determined by firms' quantity choices after uncertainty is realized. In the absence of uncertainty in ε and τ, the price vector p is then a constant. When choosing their production techniques, firms do not need to consider how it covaries with the household's stochastic discount factor, that is, the second term on the right-hand side of (A.58) is zero.